URSI COMMISSION B SCHOOL FOR YOUNG SCIENTISTS

ELECTROMAGNETIC FIELDS AND WAVES:
MATHEMATICAL MODELS AND NUMERICAL METHODS

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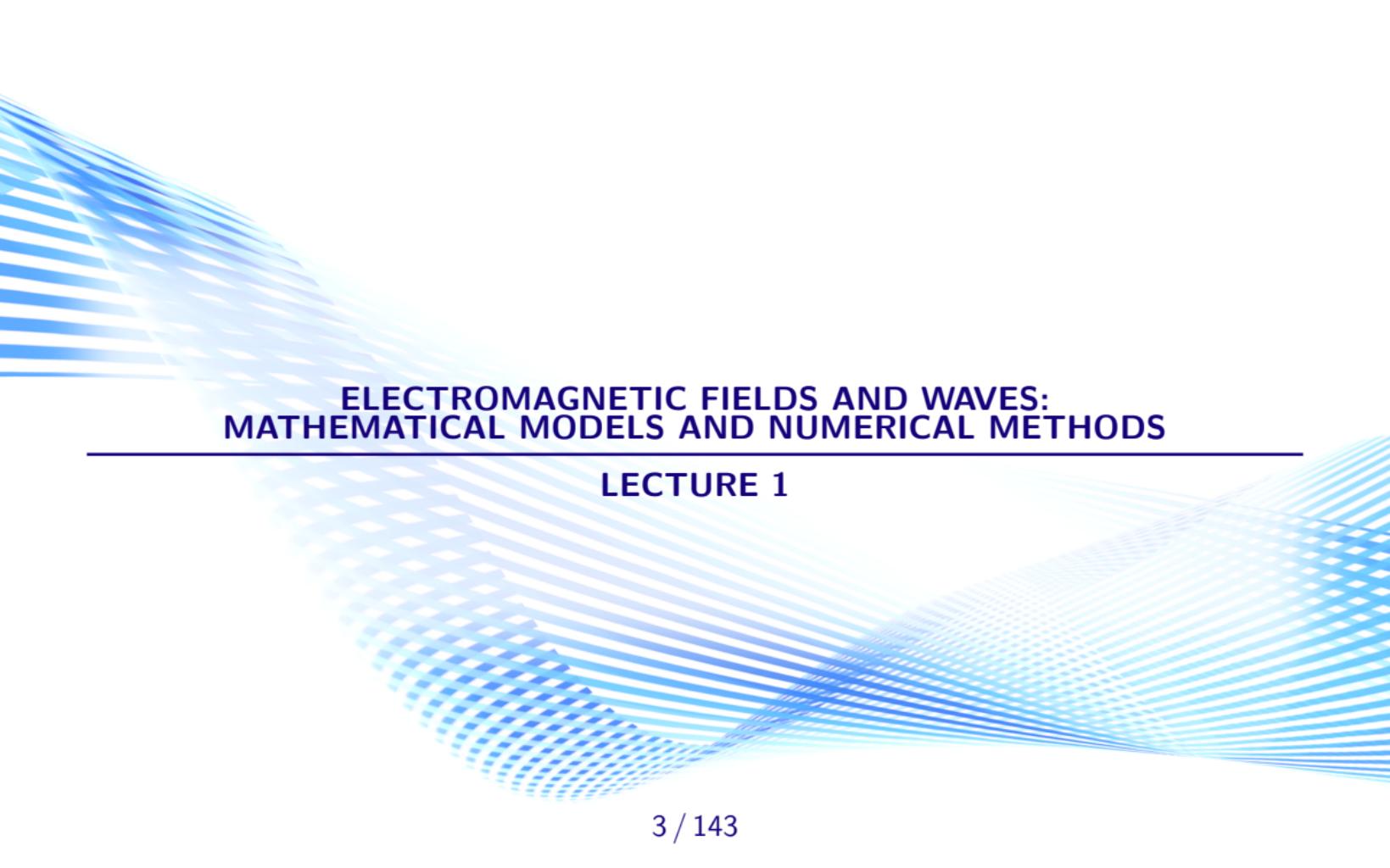
ELECTROMAGNETIC FIELDS AND WAVES: MATHEMATICAL MODELS AND NUMERICAL METHODS

LECTURE 1

LECTURE 1

DIFFERENTIAL OPERATIONS AND THEOREMS OF THE VECTOR ANALYSIS

Definition of the gradient. Vector function

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is called a gradient of (scalar) function $f(x, y, z)$.
Vector differential operator ∇ is defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Directional derivative. The directional derivative $D_\mathbf{b}f$ or $\dfrac{df}{ds}$ of a function f at a point P in the direction of a vector $\mathbf{b}$, $|\mathbf{b}| = 1$, is defined by

$$D_\mathbf{b}f = \lim_{s \to 0} \frac{f(Q) - f(P)}{s} \qquad (s = |Q - P|),$$

where Q is a variable point on the straight line C in the direction of $\mathbf{b}$.

In the Cartesian xyz-coordinates straight line C in parametric form is given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k} = \mathbf{p_0} + s\mathbf{b}$$

where $\mathbf{b}$ is a unit vector and $\mathbf{p_0}$ the position vector of P. Applying the definition it is easy to check, using the chain rule, that $D_{\mathbf{b}}f = \dfrac{df}{ds}$ is the derivative of the function $f(x(s), y(s), z(s))$ with respect to s

$$D_{\mathbf{b}}f = \frac{df}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z',$$

$$x' = \frac{dx}{ds}, \quad y' = \frac{dy}{ds}, \quad z' = \frac{dz}{ds}.$$

Differentiation gives

$$\mathbf{r}'(s) = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k} = \mathbf{b},$$

that is

$$D_{\mathbf{b}}f = \frac{df}{ds} = \mathbf{b} \cdot \mathrm{grad}\, f$$

($\mathbf{b}$ is a unit vector, $|\mathbf{b}| = 1$), or

$$D_{\mathbf{a}}f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|}\mathbf{a} \cdot \mathrm{grad}\, f$$

where $\mathbf{a} \neq 0$ is a vector of any length).

Example 1

Find the directional derivative of

$$f(x, y, z) = 2x^2 + 3y^2$$

at $P : (2, 1)$ *in the direction of* $\mathbf{a} = \mathbf{i} = [1, 0]$.

Solution.

$$f(x, y) = 2x^2 + 3y^2; \quad \frac{\partial f}{\partial x} = 4x, \quad \frac{\partial f}{\partial y} = 6y,$$

$$\operatorname{grad} f = 4x\mathbf{i} + 6y\mathbf{j}.$$

At the point $P : (2, 1)$

$$\operatorname{grad} f = 8\mathbf{i} + 6\mathbf{j} = [8, 6].$$

Since $|\mathbf{a}| = |[1, 0]| = 1$, we obtain

$$D_{\mathbf{a}} f = \frac{df}{ds} = \mathbf{i} \cdot (8\mathbf{i} + 6\mathbf{j}) = [1, 0] \cdot [8, 6] = 1 \cdot 8 + 0 \cdot 6 = 8.$$
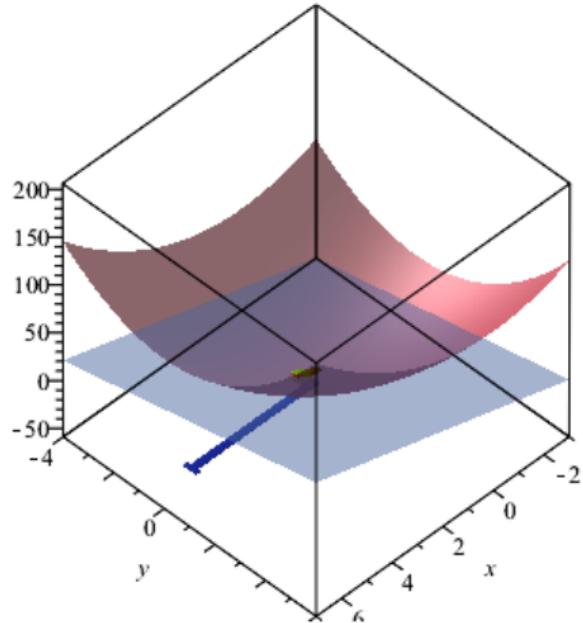
At $(x, y) = (2, 1)$, the directional derivative of $f(x, y) = 2x^2 + 3y^2$, computed in the direction i. The direction vector is projected onto the tangent plane at the evaluation point.

Theorem 1

grad f points in the direction of the maximum increase of f.

Proof. From the definition of the scalar product we have

$$D_{\mathbf{b}}f = \mathbf{b} \cdot \operatorname{grad} f = |\mathbf{b}||\operatorname{grad} f|\cos\gamma = |\operatorname{grad} f|\cos\gamma \qquad (|\mathbf{b}| = 1).$$

where γ is the angle between $\mathbf{b}$ and $\operatorname{grad} f$. Directional derivative $D_{\mathbf{b}}f$ is maximum or minimum when $\cos\gamma = 1$, $\gamma = 0$, or, respectively $\cos\gamma = -1$, $\gamma = \pi$, that is if $\mathbf{b}$ is parallel to $\operatorname{grad} f$ or, respectively $-\operatorname{grad} f$. Thus, the following statement holds.

Theorem 2

Let $f(x, y, z) = f(P)$ be a differentiable function. Then directional derivative $D_{\mathbf{b}}f$ is
(i) maximal in the direction

$$\mathbf{b} = \frac{grad\, f}{|grad\, f|}$$

and has the form

$$D_{\mathbf{b}}f = |grad\, f|;$$

(ii) minimal in the direction

$$\mathbf{b} = -\frac{grad\, f}{|grad\, f|}$$

and has the form

$$D_{\mathbf{b}}f = -|grad\, f|.$$

($grad\, f \neq 0$).

Surface normal vector. Let S be a surface represented by

$$f(x, y, z) = c = const,$$

where f is a differentiable function.

Theorem 3

If $f(x, y, z) \in C^1$) is a differentiable function and $\operatorname{grad} f \neq 0$ then $\operatorname{grad} f$ is a surface normal vector to the surface $f(x, y, z) = C$.

Proof. Let C be a curve on S through a point P of S. As a curve in space, C has a representation

$$\mathbf{r}(t) = \mathbf{v}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

If C lies on surface S, the components of $\mathbf{r}(t)$ must satisfy $f(x, y, z) = C$, that is,

$$f(x(t), y(t), z(t)) = c.$$

A tangent vector to C is

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k};$$

the tangent vectors of all curves on S passing through P will generally form a plane called the *tangent plane* of S at P. The normal to this plane (a straight line through P perpendicular to the tangent plane) is called the *surface normal* to S at P. A vector in the direction of the surface normal is called a *surface normal vector* of S at P.

We can obtain such a vector by differentiating $f(x(t), y(t), z(t)) = c$ with respect to t. By the chain rule,

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = \mathsf{grad}\, f \cdot \mathbf{r}'(t) = 0,$$

where

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}, \quad z' = \frac{dz}{dt}.$$

Hence $\mathsf{grad}\, f$ is orthogonal to all the vectors $\mathbf{r}'$ in the tangent plane, so that it is a normal vector of S at P.

Example 2

Find a unit normal vector **n** *of the cone of revolution*
$z^2 = 4(x^2 + y^2)$ *at the point* $P : (1, 0, 2)$.

Solution.
The cone is the level surface $z^2 = 4(x^2 + y^2)$, or

$$f(x, y, z) = 4x^2 + 4y^2 - z^2 = 0,$$

so that we have the equation of the cone as a level surface with
$c = 0$. The partial derivatives are

$$\frac{\partial f}{\partial x} = 8x, \quad \frac{\partial f}{\partial y} = 8y, \quad \frac{\partial f}{\partial z} = -2z,$$

and the gradient is $\operatorname{grad} f = 8x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$. At the point
$P : (1, 0, 2)$ $\operatorname{grad} f = 8\mathbf{i} - 4\mathbf{k} = [8, 0, -4]$. We have $|\operatorname{grad} f| = \sqrt{64 + 16} = \sqrt{80}$. The unit normal vector of the cone at P is

$$\mathbf{n} = \frac{1}{|\operatorname{grad} f|}\operatorname{grad} f = \frac{1}{\sqrt{80}}(8\mathbf{i} - 4\mathbf{k}) = \frac{1}{4\sqrt{5}}4(2\mathbf{i} - \mathbf{k}) = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{k}.$$
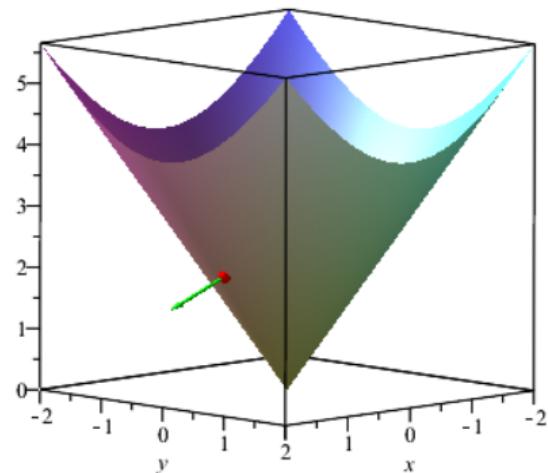
At $(x, y, z) = (1, 0, 2)$, the unit normal vector of
$f(x, y, z) = 4(x^2 + y^2) - z^2$, computed.

Definition of divergence. Let

$$\mathbf{v}(x, y, z) = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}$$

be a differentiable vector function. The (scalar) function

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the divergence of $\mathbf{v}$ or the divergence of the vector field defined by $\mathbf{v}$.
Define the vector differential operator ∇ by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Then we can write the divergence as the scalar product

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Example 3

The vector function

$$\mathbf{p} = -c \left(\frac{x - x_0}{r^3} \mathbf{i} + \frac{y - y_0}{r^3} \mathbf{j} + \frac{z - z_0}{r^3} \mathbf{k} \right),$$

where

$$\mathbf{r} = [x - x_0, y - y_0, z - z_0] = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

and

$$r = |\mathbf{r}| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2},$$

describes the gravitational force (gravitational field).

Solution.
We have

$$\frac{\partial}{\partial x} \left(\frac{1}{r} \right) = \frac{-2(x - x_0)}{2[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]} = -\frac{x - x_0}{r^3},$$

and similarly

$$\frac{\partial}{\partial y} \left(\frac{1}{r} \right) = -\frac{y - y_0}{r^3}, \quad \frac{\partial}{\partial z} \left(\frac{1}{r} \right) = -\frac{z - z_0}{r^3}.$$

Then $\mathbf{p}$ is the gradient of the function

$$f(x, y, z) = \frac{c}{r} \qquad (r > 0):$$

$$\mathbf{p} = \operatorname{grad} f = \frac{\partial}{\partial x}\left(\frac{c}{r}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{c}{r}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{c}{r}\right)\mathbf{k}$$

A vector field $\mathbf{p}$ is said to be a gradient of f if $\mathbf{p} = \operatorname{grad} f$; function f is called a scalar potential of $\mathbf{p}$. In the example above f is a scalar potential of the gravitational field.

Finding the second partial derivative using the chain rule with respect to x, y, z, we obtain

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(x - x_0)^2}{r^5},$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(y - y_0)^2}{r^5},$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = -\frac{1}{r^3} + \frac{3(z - z_0)^2}{r^5}.$$

By adding the righthand and lefthand sides, one can show that the potential f satisfies the Laplace equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

so that

$$\text{div}\,\mathbf{p} = \text{div}\,(\text{grad}\,f) = \nabla^2 f = 0.$$

Definition of rotation. Let x, y, z be a positive oriented Cartesian coordinate system and

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k}$$

a differentiable vector function. Then the vector function

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)\mathbf{k}$$

is called rotation (or curl) of vector field $\mathbf{v}$.

Example 4

Let x, y, z be a positive oriented Cartesian coordinate system. Find curl of the vector field

$$\mathbf{v}(x, y, z) = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}.$$

Solution. The curl of $\mathbf{v}$ is calculated according to

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3xz & z \end{vmatrix} =$$

$$\left(\frac{\partial z}{\partial y} - \frac{\partial(3xz)}{\partial z} \right) \mathbf{i} + \left(\frac{\partial(yz)}{\partial z} - \frac{\partial z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial(3xz)}{\partial x} - \frac{\partial(yz)}{\partial y} \right) \mathbf{k} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

Theorem 4

For any twice continuously differentiable scalar function f,

$$curl\,(grad\,f) = \mathbf{0}. \tag{1}$$

The potential (or conservative) field is called rotation-free.

Proof.

$$\mathrm{curl}\,(\mathrm{grad}\,f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} =$$

$$\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k} = (f_{zy} - f_{yz})\mathbf{i} + (f_{xz} - f_{zx})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = \mathbf{0}.$$

Theorem 5

For any twice continuously differentiable vector function $\mathbf{v}$,

$$div\,(curl\,\mathbf{v}) = 0. \tag{2}$$

The field of rotation is called divergence-free.

Proof.

$$\mathsf{div}\,(\mathsf{curl}\,\mathbf{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) =$$
$$(v_{3yx} - v_{2zx}) + (v_{1zy} - v_{3xy}) + (v_{2xz} - v_{1yz}) = 0.$$

More vector differential identities:

$$\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi.$$
$$\nabla \cdot (\phi\mathbf{F}) = \text{div}\,(\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi\nabla \cdot \mathbf{F}.$$
$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \nabla \times \mathbf{F} \cdot \mathbf{G} - \mathbf{F} \cdot \nabla \times \mathbf{G}.$$
$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}. \tag{3}$$

Formulate the divergence theorem of Gauss.

Theorem 6

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $\mathbf{F}(x, y, z)$ be a vector function that is continuous and has continuous first partial derivatives in some domain containing T. Then

$$\int\int_T\int \, div\,\mathbf{F}dV = \int_S\int \mathbf{F}\cdot\mathbf{n}dA.$$

In components

$$\int\int_T\int \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \int_S\int (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA.$$

or

$$\int\int_T\int \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \int_S\int (F_1dydz + F_2dzdx + F_3dxdy).$$

Example 5

Evaluate

$$I = \int_S \int (x^3 dydz + x^2 ydzdx + x^2 zdxdy), \tag{4}$$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ ($0 \le z \le b$) and the circular disks $z = 0$ and $z = b$ ($x^2 + y^2 \le a^2$).

Solution.

$$F_1 = x^3, \quad F_2 = x^2 y, \quad F_3 = x^2 z.$$

Hence the divergence of $\mathbf{F} = [F_1, F_2, F_3]$ is

$$\operatorname{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2.$$

The form of the surface suggests that we introduce polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{(cylindriska koordinater } r, \theta, z)$$

and

$$dxdydz = rdrd\theta dz,$$

According to Gauss's theorem, a surface integral is reduced to a triple integral if the area T is bounded by a cylindrical surface S,

$$\int_S \int (x^3 dydz + x^2 ydzdx + x^2 zdxdy) = \int \int_T \int \operatorname{div} \mathbf{F} dV = \int \int_T \int 5x^2 dxdydz =$$

$$5 \int_{z=0}^{b} \int_{r=0}^{a} \int_{\theta=0}^{2\pi} r^2 \cos^2 \theta r dr d\theta dz =$$

$$5b \int_0^a \int_0^{2\pi} r^3 \cos^2 \theta dr d\theta = 5b \frac{a^4}{4} \int_0^{2\pi} \cos^2 \theta d\theta =$$

$$5b \frac{a^4}{8} \int_0^{2\pi} (1 + 2\cos\theta) d\theta = \frac{5}{4}\pi b a^4.$$

Example 6

Evaluate

$$I = \int_S \int \mathbf{F} \cdot \mathbf{n} dA, \quad \mathbf{F} = 7x\mathbf{i} - z\mathbf{k}$$

over the sphere $S: x^2 + y^2 + z^2 = 4$. *Calculate the integral directly and using Gauss's theorem.*

Solution.
$\mathbf{F}(x, y, z) = [F_1, F_2, F_3]$ is a differentiable vector function and its components are

$$\mathbf{F} = [F_1, 0, F_3], \quad F_1 = 7x, \quad F_3 = -z.$$

The divergence of $\mathbf{F}$ is

$$\text{div}\,\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 7 + 0 - 1 = 6.$$

Accordingly,

$$I = \int \int_{T,\,\text{klot}} \int \text{div}\,\mathbf{F} dV = 6 \int \int_{T,\,\text{klot}} \int dxdydz = 6 \cdot \frac{4}{3}\pi 2^3 = 64\pi. \tag{5}$$

The surface integral of S can be calculated directly. Parametric representation of the sphere of radius 2

$$S: \ \mathbf{r}(u, v) = 2 \cos v \cos u \mathbf{i} + 2 \cos v \sin u \mathbf{j} + 2 \sin v \mathbf{k},$$

$$u, v \text{ i rectangle } R: \ 0 \le u \le 2\pi, \ -\pi/2 \le v \le \pi/2.$$

Determine the partial derivatives

$$\mathbf{r}_u = [-2 \sin u \cos v, 2 \cos v \cos u, 0],$$

$$\mathbf{r}_v = [-2 \sin v \cos u, -2 \sin v \sin u, 2 \cos v],$$

and the normal vector

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin u \cos v & 2 \cos v \cos u & 0 \\ -2 \sin v \cos u & -2 \sin v \sin u & 2 \cos v \end{vmatrix} = [4 \cos^2 v \cos u, 4 \cos^2 v \sin u, 4 \cos v \sin v].$$

On surface S,

$$x = 2 \cos v \cos u, \quad z = 2 \sin v,$$

and

$$\mathbf{F}(\mathbf{r}(u, v)) = \mathbf{F}(S) = [7x, 0, -z] = [14 \cos v \cos u, 0, -2 \sin v].$$

Then

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) = (14\cos v \cos u)4\cos^2 v \cos u + (-2\sin v)(4\cos v \sin v) = 56\cos^3 v \cos^2 u - 8\cos v \sin^2 u.$$

The parameters u, v vary in the rectangle $R: 0 \le u \le 2\pi, \ -\pi/2 \le v \le \pi/2$. Now, we can write and calculate the surface integral:

$$\int_S \int \mathbf{F} \cdot \mathbf{n} dA = \int_R \int \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) du dv = 8 \int_0^{2\pi} \int_{-\pi/2}^{-\pi/2} (7\cos^3 v \cos^2 u - \cos v \sin^2 v) du dv =$$

$$8\left\{ \frac{7}{2} \int_0^{2\pi} (1 + \cos 2u) du \int_{-\pi/2}^{\pi/2} \cos^3 v dv - 2\pi \int_{-\pi/2}^{\pi/2} \cos v \sin^2 v dv \right\} =$$

$$56\pi \int_{-\pi/2}^{\pi/2} \cos^3 v dv - 16\pi \int_{-\pi/2}^{\pi/2} \cos v dv \sin^2 v dv =$$

$$8\pi\left\{ 7 \int_{-\pi/2}^{\pi/2} (1 - \sin^2 v) d\sin v - 2 \int_{-\pi/2}^{\pi/2} dv \sin^2 v d\sin v \right\} =$$

$$8\pi\left\{ 7 \int_{-1}^{1} (1 - t^2) dt - 2 \int_{-1}^{1} t^2 dt \right\} = 8\pi[7 \cdot (2 - 2/3) - 4/3] = 8\pi \cdot 4/3 \cdot 6 = 64\pi.$$

coinciding with the value (5).
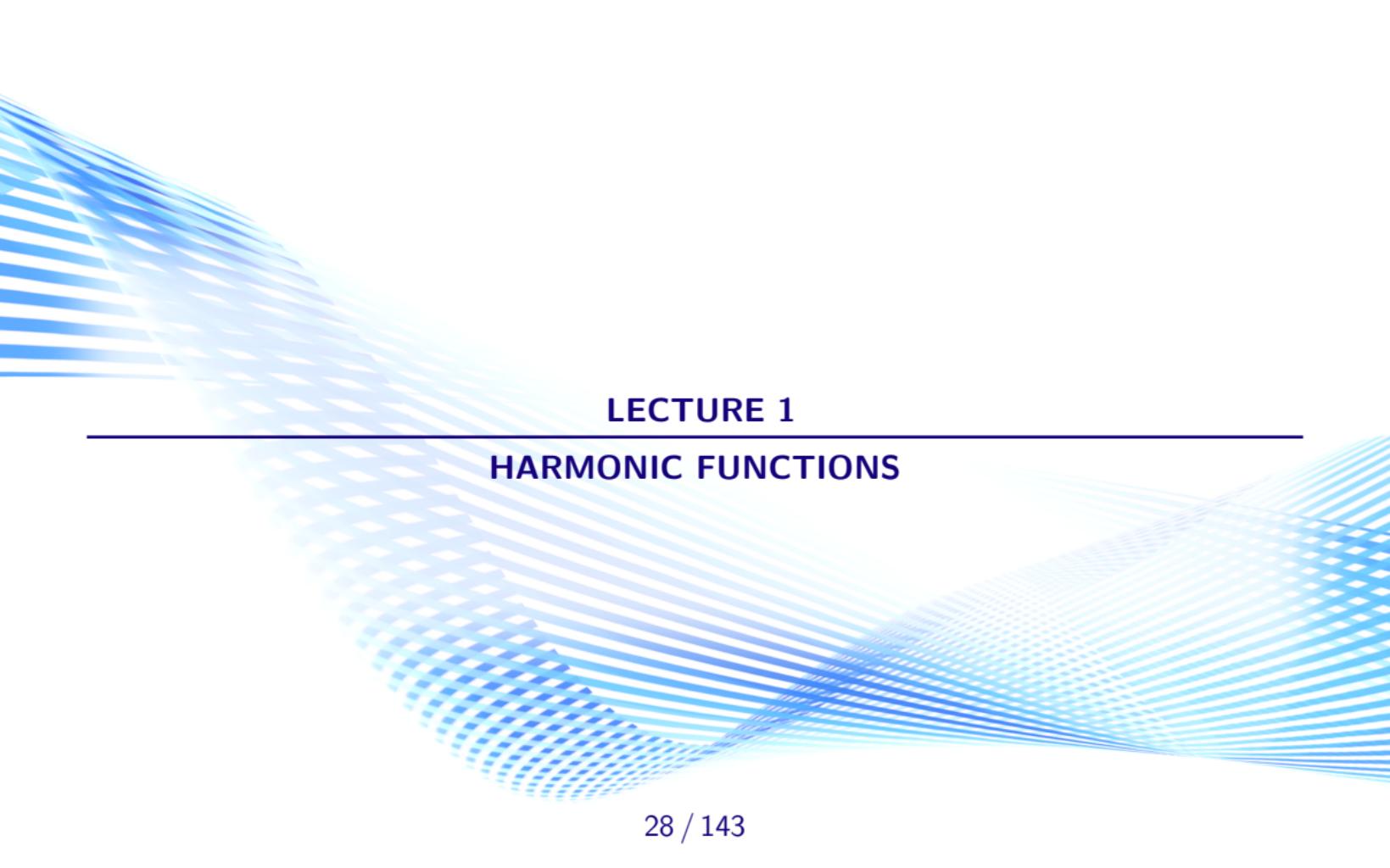
LECTURE 1

HARMONIC FUNCTIONS

A twice continuously differentiable real-valued function u defined on a domain D is called *harmonic* if it satisfies Laplace's equation

$$\Delta u = 0 \quad \text{in} \quad D, \tag{6}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{7}$$

is called *Laplace operator* (Laplacian), the function $u = u(\mathbf{x})$, and $\mathbf{x} = (x, y) \in \mathrm{R}^2$. We will also use the notation $\mathbf{y} = (x_0, y_0)$.
The function

$$\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|} \tag{8}$$

is called *the fundamental solution of the Laplace equation*. For a fixed $\mathbf{y} \in \mathrm{R}^2$, $\mathbf{y} \neq \mathbf{x}$, the function $\Phi(\mathbf{x}, \mathbf{y})$ is harmonic, i.e., satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{in} \quad D. \tag{9}$$

The proof follows by straightforward differentiation.

Let $D \in \mathrm{R}^2$ be a (two-dimensional) domain bounded by the closed smooth contour Γ and $\frac{\partial}{\partial n_y}$ denote the directional derivative in the direction of unit normal vector n_y to the boundary Γ directed into the exterior of Γ and corresponding to a point $\mathbf{y} \in \Gamma$. Then for every function u which is once continuously differentiable in the closed domain $\bar{D} = D + \Gamma$, $u \in C^1(\bar{D})$, and every function v which is twice continuously differentiable in $\bar{D}$, $v \in C^2(\bar{D})$, Green's first theorem (Green's first formula) is valid

$$\int\int_D (u\Delta v + \mathrm{grad}\, u \cdot \mathrm{grad}\, v)d\mathbf{x} = \int_\Gamma u\frac{\partial v}{\partial n_y}dl_y, \tag{10}$$

where $\cdot$ denotes the inner product of two vector-functions. For $u \in C^2(\bar{D})$ and $v \in C^2(\bar{D})$, Green's second theorem (Green's second formula) is valid

$$\int\int_D (u\Delta v - v\Delta u)d\mathbf{x} = \int_\Gamma \left(u\frac{\partial v}{\partial n_y} - v\frac{\partial u}{\partial n_y}\right)dl_y, \tag{11}$$

Let a twice continuously differentiable function $u \in C^2(\bar{D})$ be harmonic in the domain D. Then *Green's third theorem (Green's third formula)* is valid

$$u(\mathbf{x}) = \int_\Gamma \left(\Phi(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial n_y} - u(\mathbf{y})\frac{\partial \Phi(\mathbf{x},\mathbf{y})}{\partial n_y}\right)dl_y, \quad \mathbf{x} \in D. \tag{12}$$

Formulate *the interior Dirichlet problem*: find a function u that is harmonic in a domain D bounded by the closed smooth contour Γ, continuous in $\bar{D} = D \cup \Gamma$ and satisfies the Dirichlet boundary condition:

$$\Delta\, u = 0 \quad \text{in} \quad D, \tag{13}$$

$$u|_\Gamma = -f, \tag{14}$$

where f is a given continuous function.

Formulate *the interior Neumann problem*: find a function u that is harmonic in a domain D bounded by the closed smooth contour Γ, continuous in $\bar{D} = D \cup \Gamma$ and satisfies the Neumann boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_\Gamma = -g, \tag{15}$$

where g is a given continuous function.

Theorem 7

The interior Dirichlet problem has at most one solution.

Theorem 8

Two solutions of the interior Neumann problem can differ only by a constant. The exterior Neumann problem has at most one solution.

In the theory of BVPs, the integrals

$$u(\mathbf{x}) = \int\limits_C E(\mathbf{x}, \mathbf{y})\xi(\mathbf{y})dl_y, \quad v(\mathbf{x}) = \int\limits_C \frac{\partial}{\partial \mathbf{n}_y} E(\mathbf{x}, \mathbf{y})\eta(\mathbf{y})dl_y \tag{16}$$

are called *the potentials*. Here, $\mathbf{x} = (x, y)$, $\mathbf{y} = (x_0, y_0) \in \mathrm{R}^2$; $E(\mathbf{x}, \mathbf{y})$ is the fundamental solution of a second-order elliptic differential operator;

$$\frac{\partial}{\partial \mathbf{n}_y} = \frac{\partial}{\partial n_y}$$

is the normal derivative at the point $\mathbf{y}$ of the closed piecewise smooth boundary C of a domain in R^2; and $\xi(\mathbf{y})$ and $\eta(\mathbf{y})$ are sufficiently smooth functions defined on C. In the case of Laplace operator Δu,

$$E(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}. \tag{17}$$

In the case of the Helmholtz operator $L(k) = \Delta + k^2$, one can take $E(\mathbf{x}, \mathbf{y})$ in the form

$$E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x} - \mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|} + h(k|\mathbf{x} - \mathbf{y}|), \tag{18}$$

where $H_0^{(1)}(z) = -4i\Phi(z) + \tilde{h}(z)$ is the Hankel function of the first kind and zero order (one of the so-called cylindrical functions) and $\Phi(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}$ is the kernel of the two-dimensional single layer potential; $\tilde{h}(z)$ and $h(z)$ are continuously differentiable and their second derivatives have a logarithmic singularity.

Theorem 9

Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Then the kernel of the double-layer potential

$$V(\mathbf{x}, \mathbf{y}) = \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y}, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}, \tag{19}$$

is a continuous function on Γ for $\mathbf{x}, \mathbf{y} \in \Gamma$.

Gauss formula Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. For the double-layer potential with a constant density

$$v^0(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} dl_y, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}, \tag{20}$$

where the (exterior) unit normal vector n of Γ is directed into the exterior domain $\mathrm{R}^2 \setminus \bar{D}$, we have

$$\begin{aligned}
v^0(\mathbf{x}) &= -1, \quad \mathbf{x} \in D, \\
v^0(\mathbf{x}) &= -\frac{1}{2}, \quad \mathbf{x} \in \Gamma, \\
v^0(\mathbf{x}) &= 0, \quad \mathbf{x} \in \mathrm{R}^2 \setminus \bar{D}.
\end{aligned} \tag{21}$$

Corollary. Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Introduce the single-layer potential with a constant density

$$u^0(\mathbf{x}) = \int\limits_{\Gamma} \Phi(\mathbf{x}, \mathbf{y}) dl_y, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}. \tag{22}$$

For the normal derivative of this single-layer potential

$$\frac{\partial u^0(\mathbf{x})}{\partial n_x} = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_x} dl_y, \tag{23}$$

where the (exterior) unit normal vector n_x of Γ is directed into the exterior domain $\mathrm{R}^2 \setminus \bar{D}$, we have

$$\begin{aligned}
\frac{\partial u^0(\mathbf{x})}{\partial n_x} &= 1, \quad \mathbf{x} \in D, \\
\frac{\partial u^0(\mathbf{x})}{\partial n_x} &= \frac{1}{2}, \quad \mathbf{x} \in \Gamma, \\
\frac{\partial u^0(\mathbf{x})}{\partial n_x} &= 0, \quad \mathbf{x} \in \mathrm{R}^2 \setminus \bar{D}.
\end{aligned} \tag{24}$$

Theorem 10

Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential

$$v(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}, \tag{25}$$

with a continuous density φ can be continuously extended from D to $\bar{D}$ and from $R^2 \setminus \bar{D}$ to $R^2 \setminus D$ with the limiting values on Γ

$$v_{\pm}(\mathbf{x}') = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}', \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y \pm \frac{1}{2} \varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{26}$$

or

$$v_{\pm}(\mathbf{x}') = v(\mathbf{x}') \pm \frac{1}{2} \varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{27}$$

where

$$v_{\pm}(\mathbf{x}') = \lim_{h \to \pm 0} v(\mathbf{x} + h n_{x'}). \tag{28}$$

Corollary. Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. Introduce the single-layer potential

$$u(\mathbf{x}) = \int\limits_{\Gamma} \Phi(\mathbf{x}, \mathbf{y})\varphi(\mathbf{y})dl_y, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}. \tag{29}$$

with a continuous density φ. The normal derivative of this single-layer potential

$$\frac{\partial u(\mathbf{x})}{\partial n_x} = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_x}\varphi(\mathbf{y})dl_y \tag{30}$$

can be continuously extended from D to $\bar{D}$ and from $\mathrm{R}^2 \setminus \bar{D}$ to $\mathrm{R}^2 \setminus D$ with the limiting values on Γ

$$\frac{\partial u(\mathbf{x}')}{\partial n_x}\bigg|_{\pm} = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}', \mathbf{y})}{\partial n_{x'}}\varphi(\mathbf{y})dl_y \mp \frac{1}{2}\varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{31}$$

or

$$\frac{\partial u(\mathbf{x}')}{\partial n_x}\bigg|_{\pm} = \frac{\partial u(\mathbf{x}')}{\partial n_x} \mp \frac{1}{2}\varphi(\mathbf{x}'), \quad \mathbf{x}' \in \Gamma, \tag{32}$$

where

$$\frac{\partial u(\mathbf{x}')}{\partial n_{x'}} = \lim_{h \to \pm 0} n_{x'} \cdot \mathsf{grad}\, v(\mathbf{x}' + h n_{x'}). \tag{33}$$

LECTURE 1: HARMONIC FUNCTIONS

Let $S_\Pi(\Gamma) \in \mathrm{R}^2$ be a domain bounded by the closed piecewise smooth contour Γ. We assume that a rectilinear interval Γ_0 is a subset of Γ, so that $\Gamma_0 = \{\mathbf{x} : y = 0, x \in [a, b]\}$.

Let us say that functions $U_l(\mathbf{x})$ are *the generalized single layer (SLP)* $(l = 1)$ *or double layer (DLP)* $(l = 2)$ *potentials* if

$$U_l(\mathbf{x}) = \int\limits_\Gamma K_l(\mathbf{x}, t)_l(t)dt, \quad \mathbf{x} = (x, y) \in S_\Pi(\Gamma),$$

where

$$K_l(\mathbf{x}, t) = g_l(\mathbf{x}, t) + F_l(\mathbf{x}, t) \quad (l = 1, 2),$$

$$g_1(\mathbf{x}, t) = g(x, y^0) = \frac{1}{\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}^0|}, \quad g_2(\mathbf{x}, t) = \frac{\partial}{\partial y_0} g(\mathbf{x}, \mathbf{y}^0) \quad [\mathbf{y}^0 = (t, 0)],$$

$F_{1,2}$ are smooth functions, and we shall assume that for every closed domain $S_{0\Pi}(\Gamma) \subset S_\Pi(\Gamma)$, the following conditions hold

i) $F_1(\mathbf{x}, t)$ is once continuously differentiable with respect to the
    variables of $\mathbf{x}$ and continuous in t;
ii) $F_2(\mathbf{x}, t)$ and

$$F_2^1(\mathbf{x}, t) = \frac{\partial}{\partial y} \int\limits_q^t F_2(x, s)ds, \quad q \in \mathrm{R}^1,$$

are continuous.

Introduce integral operators K_0 and K_1 acting in the space $C(\Gamma)$ of continuous functions defined on contour Γ

$$K_0(\mathbf{x}) = 2 \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \mathbf{x} \in \Gamma \tag{34}$$

and

$$K_1(\mathbf{x}) = 2 \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_x} \psi(\mathbf{y}) dl_y, \quad \mathbf{x} \in \Gamma. \tag{35}$$

Theorem 11

The operators $I - K_0$ and $I - K_1$ have trivial nullspaces

$$N(I - K_0) = \{0\}, \quad N(I - K_1) = \{0\},$$

The nullspaces of operators $I + K_0$ and $I + K_1$ have dimension one and

$$N(I + K_0) = \text{span}\,\{1\}, \quad N(I + K_1) = \text{span}\,\{\psi_0\}$$

with

$$\int\limits_{\Gamma} \psi_0 dl_y \neq 0.$$

Theorem 12

Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential

$$v(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}, \quad \mathbf{x} \in D, \tag{36}$$

with a continuous density φ is a solution of the interior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(\mathbf{x}) - 2 \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y = -2f(x), \quad \mathbf{x} \in \Gamma, \tag{37}$$

where $f(x)$ is given by (14).

Theorem 13

The interior Dirichlet problem has a unique solution.

Theorem 14

Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double-layer potential

$$u(\mathbf{x}) = \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y, \quad \mathbf{x} \in R^2 \setminus \bar{D}, \tag{38}$$

with a continuous density φ is a solution of the exterior Dirichlet problem provided that φ is a solution of the integral equation

$$\varphi(\mathbf{x}) + 2 \int\limits_{\Gamma} \frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial n_y} \varphi(\mathbf{y}) dl_y = 2f(x), \quad \mathbf{x} \in \Gamma. \tag{39}$$

Here we assume that the origin is contained in D.

Theorem 15

The exterior Dirichlet problem has a unique solution.

Theorem 16

Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The single-layer potential

$$u(\mathbf{x}) = \int\limits_{\Gamma} \Phi(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})dl_y, \quad \mathbf{x} \in D, \tag{40}$$

with a continuous density ψ is a solution of the interior Neumann problem provided that ψ is a solution of the integral equation

$$\psi(\mathbf{x}) + 2\int\limits_{\Gamma} \frac{\partial\Phi(\mathbf{x}, \mathbf{y})}{\partial n_x}\psi(\mathbf{y})dl_y = 2g(x), \quad \mathbf{x} \in \Gamma. \tag{41}$$

Theorem 17

The interior Neumann problem is solvable if and only if

$$\int_{\Gamma} \psi dl_y = 0 \tag{42}$$

is satisfied.

LECTURE 1

LINE INTEGRALS. GREEN'S FORMULA. SURFACE INTEGRALS

Curves in a parametric form and line integrals. Let xyz be a Cartesian coordinate system in space. We write a spatial curve C using a parametric representation

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad (t \in I), \tag{43}$$

where variable t is a parameter.

As far as a line integral over a spatial curve C is concerned, C is called the path of integration. The path of integration with spatial endpoints A to B goes from A to B (has a certain direction) so that $A := \mathbf{r}(a)$ is its initial point and $B := \mathbf{r}(b)$ is its terminal point. C is now oriented. The direction from A to B, in which t increases is called the positive direction on C. Points A and B may coincide, then C is called a closed path.

Definition of line integral. If C is an oriented curve in a parametric form

$$P = P(t) \quad (x = x(t), \quad y = y(t), \quad z = z(t)) \quad t \in I = (t_0, t_1), \quad t: \ t_0 \to t_1, \tag{44}$$

and $f(P)$ and $g(P)$ are real (or complex) function defined on C, the *line integral of a scalar function* is defined as

$$\int_C f(P) dg(P) = \int_{t=t_0}^{t=t_1} f(P(t)) dg(P(t)), \tag{45}$$

(if the right-hand side in the equality specifying the integral exists).

A *line integral* of a vector function $\mathbf{F}(\mathbf{r})$ over a curve C is defined by

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt,$$

or componentwise

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z') dt \quad (' = d/dt).$$

Example 7

Find the value of the line integral when $\mathbf{F}(\mathbf{r}) = [-y, -xy]$ and C is a circular arc from $(1,0)$ to $(0,1)$.

Solution. We may represent C by

$$\mathbf{r}(t) = [\cos t, \sin t] = \cos t\mathbf{i} + \sin t\mathbf{j}, \tag{46}$$

and

$$\mathbf{r}(t) = [\cos t, \sin t], \quad t: \ 0 \to \pi/2.$$

The parameter interval is $I = (t_0, t_1)$ with the initial point $t_0 = 0$ and endpoint $t_1 = \pi/2$. In such an orientation,

$$P(0) = (\cos 0, b\sin 0) = (1,0)$$

is the initial point and

$$P(\pi/2) = (a\cos \pi/2, b\sin \pi/2) = (0,1)$$

is the endpoint.

We have $x = \cos t$, $y = \sin t$ and can write vector function $\mathbf{F}(\mathbf{r})$ on the unit circle

$$\mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j} = [-\sin t, -\cos t\sin t] = -\sin t\mathbf{i} - \cos t\sin t\mathbf{j}.$$

Determine

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

and calculate the line integral:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{\pi/2} (-\sin t\mathbf{i} - \cos t \sin t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})dt =$$

$$\int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)dt = \int_0^{\pi/2} [(1/2)(1 - \cos 2t) - \cos^2 t \sin t]dt =$$

$$(1/2)\int_0^{\pi/2} [(1 - \cos 2t)dt + \int_0^{\pi/2} \cos^2 td\cos t = \frac{\pi}{4} - \frac{1}{3}.$$

Example 8

Find the line integral for $\mathbf{F}(\mathbf{r}) = [5z, xy, x^2z]$ when curves C_1 and C_2 have the same initial point $A : (0, 0, 0)$ and endpoint $B : (1, 1, 1)$, C_1 is an interval of the straight line

$$\mathbf{r}_1(t) = [t, t, t] = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1,$$

and C_2 is a parabola

$$\mathbf{r}_2(t) = [t, t, t^2] = t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}, \quad 0 \le t \le 1.$$

Solution. We have

$$\mathbf{F}(\mathbf{r}_1(t)) = 5t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{F}(\mathbf{r}_2(t)) = 5t^2\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k},$$

$$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j} + \mathbf{j}, \quad \mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 2t\mathbf{j}.$$

Then we can calculate the line integral over C_1

$$\int_{C_1} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)dt = \int_0^1 (5t + t^2 + t^3)dt = \frac{5}{2} + \frac{1}{3} + \frac{1}{4} = \frac{37}{12}.$$

The line integral over C_2 is

$$\int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t)dt = \int_0^1 (5t^2 + t^2 + 2t^5)dt = \frac{5}{3} + \frac{1}{3} + \frac{2}{6} = \frac{28}{12}.$$

Thus we have got two different values.

Theorem 18

The line integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz),$$

where F_1, F_2, F_3 are continuous functions on a domain D in space, is path independent in D, if and only if $\mathbf{F} = [F_1, F_2, F_3]$ is the gradient of a function $f = f(x, y, z)$ in D :

$$\mathbf{F} = grad\, f;$$

with the components

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}.$$

If $\mathbf{F}$ is the gradient field and f is a scalar potential of $\mathbf{F}$ then the line integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A),$$

where A is the initial point and B the endpoint of C.

Example 9

Show that the integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (2xdx + 2ydy + 4zdz)$$

is path independent in any domain in space and find its value if integration is performed from $A : (0,0,0)$ to $B : (2,2,2)$.

Solution. We have

$$\mathbf{F} = [2x, 2y, 4z] = 2x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k} = \mathsf{grad}\, f,$$

and it is easy to check that

$$f(x,y,z) = x^2 + y^2 + 2z^2.$$

According to Theorem 18, the line integral is path independent in any domain in space. To find its value, we choose the convenient straight path

$$\mathbf{r}(t) = [t, t, t] = t(\mathbf{i} + \mathbf{j} + \mathbf{k}), \quad 0 \le t \le 2.$$

Let $A : (0,0,0)$, $t = 0$, be the initial point and $B : (2,2,2)$, $t = 2$ the endpoint. Then we get

$$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + \mathbf{j}.$$

$$\mathbf{F}(\mathbf{r}) \cdot \mathbf{r}' = 2t + 2t + 4t = 8t$$

and

$$\int_C (2xdx + 2ydy + 4zdz) = \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^2 8tdt = 16.$$

According to Theorem 18,

$$\int_C \mathbf{F}(\mathbf{r})d\mathbf{r} = f(2,2,2) - f(0,0,0) = 4 + 4 + 2 \cdot 4 - 0 = 16.$$

Theorem 19

The line integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

where F_1, F_2, F_3 are continuous functions on a domain D in space is path independent in D if and only if

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$$

along every closed path C in D.

The differential form

$$F_1 dx + F_2 dy + F_3 dz$$

is called exact in a domain D in space if it is the differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

of a differentiable function $f(x, y, z)$ everywhere in D:

$$F_1 dx + F_2 dy + F_3 dz = df,$$

where

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}.$$

Green's formula. Let C be a closed curve in xy-plane that does not intersect itself and makes just one turn in the positive direction (counterclockwise). Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\dfrac{\partial F_1}{\partial y}$ and $\dfrac{\partial F_2}{\partial x}$ everywhere in some domain R enclosed by C. Then

$$\int_R \int \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \int_C (F_1 dx + F_2 dy).$$

Here we integrate along the entire boundary C of R so that R is on the left as we advance in the direction of integration.
One can write Green's formula with the help of curl

$$\int_R \int (\text{curl } \mathbf{F}) \cdot \mathbf{k} dx dy = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Example 10

Verify Green's formula for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C being a circle $R : x^2 + y^2 = 1$.

Solution. Calculate a double integral

$$\int_R \int \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_R \int [(2y + 2) - (2y - 7)]dxdy = 9 \int_R \int dxdy = 9\pi.$$

Calculate the corresponding line integral. Circle C in the parametric form is given by

$$\mathbf{r}(t) = [\cos t, \sin t] = \cos t\mathbf{i} + \sin t\mathbf{j}.$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}.$$

On C

$$F_1 = y^2 - 7y = \sin^2 t - 7\sin t, \quad F_2 = 2xy + 2x = 2\cos t\sin t + 2\cos t,$$

and we get that the line integral in Green's formula is equal to the double integral:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} [(\sin^2 t - 7\sin t)(-\sin t) + (2\cos t\sin t + 2\cos t)\cos t]dt = 0 + 7\pi + 0 + 2\pi = 9\pi.$$

Surface integral. To define a surface integral, we take a surface S given by a parametric representation

$$\mathbf{r}(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad u, v \in R,$$

the normal vector

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0},$$

and unit normal vector

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}.$$

A surface integral of a vector function $\mathbf{F}(\mathbf{r})$ over a surface S is defined as

$$\int_S \int \mathbf{F} \cdot \mathbf{n} dA = \int_R \int \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) du dv. \tag{47}$$

Note that

$$\mathbf{n}dA = \mathbf{n}|\mathbf{N}|dudv = |\mathbf{N}|dudv,$$

and we assume that the parameters u, v belongs to a region R in the u, v-plane.
Write the equivalent expression componentwise using directional cosine:

$$\mathbf{F} = [F_1, F_2, F_3] = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k},$$

$$\mathbf{n} = [\cos\alpha, \cos\beta, \cos\gamma] = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k},$$

$$\mathbf{N} = [N_1, N_2, N_3] = N_1\mathbf{i} + N_2\mathbf{j} + N_3\mathbf{k},$$

and

$$\int_S \int \mathbf{F} \cdot \mathbf{n}dA = \int_S \int (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA =$$

$$\int_S \int (F_1 N_1 + F_2 N_2 + F_3 N_3)dudv.$$

Example 11

Evaluate a surface integral of the vector function $\mathbf{F} = [x^2, 0, 3y^2]$ over a portion of the plane

$$S: \; x + y + z = 1, \; 0 \leq x, y, z \leq 1.$$

Solution. Writing $x = u$ and $y = v$, we have $z = 1 - u - v$ and can represent S in the form

$$\mathbf{r}(u,v) = [u, v, 1 - u - v], \quad 0 \leq v \leq 1, \; 0 \leq u \leq 1 - v.$$

We have

$$\mathbf{r}_u = [1, 0, -1], \quad \mathbf{r}_v = [0, 1, -1];$$

a normal vector is

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k} = [1, 1, 1].$$

The corresponding unit normal vector

$$\mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

On surface S,

$$\mathbf{F}(\mathbf{r}(u,v)) = \mathbf{F}(S) = [u^2, 0, 3v^2] = u^2\mathbf{i} + 3v^2\mathbf{k}.$$

Hence

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) = [u^2, 0, 3v^2] \cdot [1,1,1] = u^2 + 3v^2.$$

Parameters u, v belong to triangle $R: 0 \le v \le 1, \ 0 \le u \le 1 - v$. Now we can write and calculate the flux integral:

$$\int_S \int \mathbf{F} \cdot \mathbf{n} dA = \int_R \int \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v) du dv = \int_R \int (u^2 + 3v^2) du dv =$$

$$\int_0^1 \int_0^{1-v} (u^2 + 3v^2) du dv = \int_0^1 dv \int_0^{1-v} u^2 du + 3 \int_0^1 v^2 dv \int_0^{1-v} du =$$

$$= (1/3) \int_0^1 (1-v)^3 dv + 3 \int_0^1 v^2(1-v) dv = (1/3) \int_0^1 t^3 dt + 3 \int_0^1 (v^2 - v^3) dv =$$

$$(1/3) \cdot (1/4) + 3(1/3 - 1/4) = 1/3.$$

LECTURE 1

BASIC ELECTROMAGNETIC THEORY
MAXWELLS AND HELMHOLTZ EQUATIONS

The classical macroscopic electromagnetic field is described by four three-component vector-functions $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{D}(\mathbf{r}, t)$, $\mathbf{H}(\mathbf{r}, t)$, and $\mathbf{B}(\mathbf{r}, t)$ of the position vector $\mathbf{r} = (x, y, z)$ and time t. The fundamental field vectors $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ are called *electric* and *magnetic field intensities*. $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ which will be eliminated from the description via constitutive relations are called *the electric displacement* and *magnetic induction*. The fields and sources are related by *the Maxwell equation system*

$$\frac{\partial \mathbf{D}}{\partial t} - \mathrm{rot}\,\mathbf{H} = -\mathbf{J}, \tag{48}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathrm{rot}\,\mathbf{E} = 0, \tag{49}$$

$$\mathrm{div}\,\mathbf{B} = 0, \tag{50}$$

$$\mathrm{div}\,\mathbf{D} = \rho, \tag{51}$$

written in the standard SI units.

The constitutive relations are

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{52}$$

$$\mathbf{B} = \mu \mathbf{H}, \tag{53}$$

$$\mathbf{J} = \sigma \mathbf{E}. \tag{54}$$

Here ϵ, μ, and σ, which are generally bounded functions of position (the first two are assumed positive), are permittivity, permeability, and conductivity of the medium for $\mathbf{J}$ being the conductivity current density.
In vacuum, that is, in a homogeneous medium with constant characteristics $\epsilon = \epsilon_0$, $\mu = \mu_0$, and $\sigma = 0$, the Maxwell equation system takes a simpler form

$$\mathrm{rot}\,\mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{55}$$

$$\mathrm{rot}\,\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{56}$$

$$\mathrm{div}\,\mathbf{H} = 0, \tag{57}$$

$$\mathrm{div}\,\mathbf{E} = \rho. \tag{58}$$

In the case of a homogeneous medium, it is reasonable to obtain equations for each vector $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$. To this end, assume that $\rho = 0$. Applying the operation rot to equation (48) and taking into account the constitutive relations, we have

$$\text{rotrot}\mathbf{H} = \epsilon \frac{\partial}{\partial t} \text{rot}\mathbf{E} + \sigma \text{rot}\mathbf{E}. \tag{59}$$

Using the vector differential identity $\text{rotrot}\mathbf{A} = \text{graddiv}\,\mathbf{A} - \Delta\mathbf{A}$ and taking into notice equation (49), we obtain the equation for magnetic field $\mathbf{H}$

$$\text{graddiv}\,\mathbf{H} - \Delta\mathbf{H} = -\epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t}$$

or

$$\Delta\mathbf{H} = \frac{1}{a^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} + \sigma\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \left(a^2 = \frac{1}{\epsilon\mu}\right) \tag{60}$$

because $\text{div}\,\mathbf{H} = 0$.

The same equation holds for electric field $\mathbf{E}$

$$\Delta\mathbf{E} = \frac{1}{a^2}\frac{\partial^2\mathbf{E}}{\partial t^2} + \sigma\mu\frac{\partial\mathbf{E}}{\partial t} \qquad \left(a^2 = \frac{1}{\epsilon\mu}\right). \tag{61}$$

Equations (60) or (61) hold for all field components,

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} + \sigma\mu\frac{\partial u}{\partial t}, \tag{62}$$

where u is one of the components H_x, H_y, H_z or E_x, E_y, E_z.
If the medium is nonconducting, $\sigma = 0$, then (60), (61), or (62) yield a standard wave equation

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}. \tag{63}$$

This implies that electromagnetic processes are actually *waves* that propagate in the medium with the speed $a = \frac{1}{\sqrt{\epsilon\mu}}$ (the latter holds for vacuum).

LECTURE 1: BASIC ELECTROMAGNETIC THEORY. MAXWELLS AND HELMHOLTZ EQUATIONS.

Time-periodic (time-harmonic) fields

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r})e^{-i\omega t}, \qquad \mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{-i\omega t} \tag{64}$$

constitute a very important particular case. Functions $\mathbf{E}$ and $\mathbf{H}$ are the field complex amplitudes; the quantities $\operatorname{Re}\mathbf{E}$ and $\operatorname{Re}\mathbf{H}$ have direct physical meaning.

Assuming that complex electromagnetic field (64) satisfies Maxwell equations and that the currents are also time-harmonic, $\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r})e^{-i\omega t}$, substitute (64) into (48)–(51) to obtain

$$\operatorname{rot}\mathbf{H} = -i\omega\mathbf{D} + \mathbf{J}, \tag{65}$$
$$\operatorname{rot}\mathbf{E} = i\omega\mathbf{B}, \tag{66}$$
$$\operatorname{div}\mathbf{B} = 0, \tag{67}$$
$$\operatorname{div}\mathbf{D} = \rho. \tag{68}$$

Since $\mathbf{J} = \sigma\mathbf{E}$, equation (65) can be transformed by introducing the complex permittivity

$$\epsilon' = \epsilon + i\frac{\sigma}{\omega}.$$

As a result, system (65)–(68) takes the form

$$\operatorname{rot}\mathbf{H} = -i\omega\epsilon'\mathbf{E}, \tag{69}$$
$$\operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}, \tag{70}$$
$$\operatorname{div}(\mu\mathbf{H}) = 0, \tag{71}$$
$$\operatorname{div}(\epsilon\mathbf{E}) = \rho. \tag{72}$$

In a homogeneous medium and when external currents are absent, equations (71) and (72) follow from the first two Maxwell equations (69) and (70).

Consider the simplest time-harmonic solutions to Maxwell equations in a homogeneous medium (with constant characteristics), plane electromagnetic waves. In the absence of free charges when div $\mathbf{E} = 0$, the electric field vector satisfies the equation

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{E} = \omega^2 \epsilon' \mu \mathbf{E}, \tag{73}$$

or

$$\Delta \mathbf{E} + \kappa^2 \mathbf{E} = 0, \tag{74}$$

where

$$\epsilon' = \epsilon + i\frac{\sigma}{\omega}, \quad \kappa^2 = \omega^2 \epsilon' \mu = k^2 + i\omega\mu\sigma, \quad k = \omega\sqrt{\epsilon\mu}. \tag{75}$$

In the cartesian coordinate system, equation (74) holds for every field component,

$$\Delta u + \kappa^2 u = 0, \tag{76}$$

where u is one of the components E_x, E_y, E_z.

The Helmholtz equation (76) has a solution in the form of a plane wave; componentwise,

$$E_\alpha = E_\alpha^0 e^{i(\kappa_x x + \kappa_y y + \kappa_z z)}, \quad \kappa_x^2 + \kappa_y^2 + \kappa_z^2 = \kappa^2 \quad (\alpha = x, y, z). \tag{77}$$

Here κ is called the wave propagation constant. Therefore, the vector Helmholtz equation (74) has a solution

$$\mathbf{E} = \mathbf{E}_0 e^{i(\kappa_x x + \kappa_y y + \kappa_z z)} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{78}$$

where the vectors

$$\mathbf{k} = (\kappa_x, \kappa_y, \kappa_z), \quad \mathbf{r} = (x, y, z), \quad \mathbf{E}_0 = const. \tag{79}$$

Since div $\mathbf{E} = 0$, we have

$$\text{div}\,\mathbf{E} = \text{div}\,(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}) = i e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{k} \cdot \mathbf{E}_0 = 0.$$

Thus, $\mathbf{k} \cdot \mathbf{E}_0 = 0$ so that the direction of vector $\mathbf{E}$ is orthogonal to the direction of the plane wave propagation governed by vector $\mathbf{k}$.

Vectors $\mathbf{E}$ and $\mathbf{H}$ are coupled by the relation

$$\text{rot}\mathbf{E} = i\omega\mu\mathbf{H}. \tag{80}$$

Since

$$\text{rot}\,(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}) = [\text{grad}\,e^{i\mathbf{k}\cdot\mathbf{r}},\,\mathbf{E}_0],$$

we have

$$\sqrt{\epsilon'}[\mathbf{k}_0,\,\mathbf{E}_0] = \sqrt{\mu}\mathbf{H}_0, \tag{81}$$

where $\mathbf{k}_0 = \mathbf{k}/|\mathbf{k}|$ is the unit vector in the direction of the wave propagation. Thus, vectors $\mathbf{E}$ and $\mathbf{H}$ are not only orthogonal to the direction of the wave propagation but also mutually orthogonal:

$$\mathbf{E}\cdot\mathbf{H} = 0, \quad \mathbf{E}\cdot\mathbf{k} = 0, \quad \mathbf{H}\cdot\mathbf{k} = 0. \tag{82}$$

We see that the Maxwell equations have a solution in the form of a plane electromagnetic wave

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{83}$$

where

$$\sqrt{\epsilon'}[\mathbf{k}_0,\,\mathbf{E}] = \sqrt{\mu}\mathbf{H}, \quad \sqrt{\mu}[\mathbf{k}_0,\,\mathbf{H}] = -\sqrt{\epsilon'}\mathbf{E}, \tag{84}$$

Introduce the dimensionless variables and parameters

$$k_0 x \to x, \quad \sqrt{\mu_0/\varepsilon_0} \mathbf{H} \to \mathbf{H}, \quad \mathbf{E} \to \mathbf{E}, \quad k_0^2 = \varepsilon_0 \mu_0 \omega^2,$$

where ε_0 and μ_0 are permittivity and permeability of vacuum. Propagation of electromagnetic waves along a tube (a waveguide) with cross section Ω (a 2-D domain bounded by smooth curve Γ) parallel to the x_3-axis in the cartesian coordinate system $x_1,\ x_2,\ x_3,\ \mathbf{x} = (x_1,\ x_2,\ x_3)$, is described by the homogeneous system of Maxwell equations (written in the normalized form) with the electric and magnetic field dependence $e^{i\gamma x_3}$ on longitudinal coordinate x_3 (the time factor $e^{i\omega t}$ is omitted):

$$\begin{aligned}
&\text{rot}\,\mathbf{E} = -i\mathbf{H}, \quad \mathbf{x} \in \Sigma, \\
&\text{rot}\,\mathbf{H} = i\varepsilon\mathbf{E}, \\
&\mathbf{E}\,(\mathbf{x}) = \left(E_1\,(\mathbf{x}')\,\mathbf{e}_1 + E_2\,(\mathbf{x}')\,\mathbf{e}_2 + E_3\,(\mathbf{x}')\,\mathbf{e}_3\right) e^{i\gamma x_3}, \\
&\mathbf{H}\,(\mathbf{x}) = \left(H_1\,(\mathbf{x}')\,\mathbf{e}_1 + H_2\,(\mathbf{x}')\,\mathbf{e}_2 + H_3\,(\mathbf{x}')\,\mathbf{e}_3\right) e^{i\gamma x_3}, \\
&\mathbf{x}' = (x_1,\ x_2),
\end{aligned} \tag{85}$$

with the boundary conditions for the tangential electric field components on the perfectly conducting surfaces

$$\mathbf{E}_\tau|_M = 0, \tag{86}$$

Write system of Maxwell equations (85) componentwise

$$\frac{\partial H_3}{\partial x_2} - i\gamma H_2 = i\varepsilon E_1, \quad \frac{\partial E_3}{\partial x_2} - i\gamma E_2 = -iH_1, \quad i\gamma H_1 - \frac{\partial H_3}{\partial x_1} = i\varepsilon E_2,$$

$$i\gamma E_1 - \frac{\partial E_3}{\partial x_1} = -iH_2, \quad \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} = i\varepsilon E_3, \quad \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} = -iH_3,$$

and express functions E_1, H_1, E_2, and H_2 via E_3 and H_3 from the first, second, fourth, and fifth equalities, denoting $\tilde{k}^2 = \varepsilon - \gamma^2$,

$$E_1 = \frac{i}{\tilde{k}^2}\left(\gamma\frac{\partial E_3}{\partial x_1} - \frac{\partial H_3}{\partial x_2}\right), \quad E_2 = \frac{i}{\tilde{k}^2}\left(\gamma\frac{\partial E_3}{\partial x_2} + \frac{\partial H_3}{\partial x_1}\right), \tag{87}$$

$$H_1 = \frac{i}{\tilde{k}^2}\left(\varepsilon\frac{\partial E_3}{\partial x_2} + \gamma\frac{\partial H_3}{\partial x_1}\right), \quad H_2 = \frac{i}{\tilde{k}^2}\left(-\varepsilon\frac{\partial E_3}{\partial x_1} + \gamma\frac{\partial H_3}{\partial x_2}\right).$$

Note that this representation is possible if $\gamma^2 \neq \varepsilon_1$ and $\gamma^2 \neq \varepsilon_2$.
It follows from (87) that the field of a normal wave can be expressed via two scalar functions

$$\Pi(x_1, x_2) = E_3(x_1, x_2), \quad \Psi(x_1, x_2) = H_3(x_1, x_2).$$

If to look for particular solutions with $E_3 \equiv 0$ then we have a separate problem for the set of component functions $[E_1, E_2, H_3]$, $[H_1, H_2, 0]$ which are called *TE-waves* (transverse electric) or the case of H-polarization. For particular solutions with $H_3 \equiv 0$ we have a problem for the set of component functions $[H_1, H_2, E_3]$, $[E_1, E_2, 0]$ called *TM-waves* (transverse magnetic) or the case of E-polarization. These two cases constitute two fundamental polarizations of the electromagnetic field associated with a given direction of propagation. For $\gamma = 0$ when we consider fields independent of one of the coordinates (x_3) we have two separate problems for the sets of component functions $[E_1, E_2, H_3]$, TE-(H)polarization, and $[H_1, H_2, E_3]$, TM-(E)polarization. Thus the problem on normal waves is reduced to boundary eigenvalue problems for functions Π and Ψ. Namely, from (85) and (86) we have the following eigenvalue problem on normal waves in a waveguide with homogeneous filling: to find $\gamma \in C$, called eigenvalues of normal waves such that there exist nontrivial solutions of the Helmholtz equations

$$\Delta \Pi + \tilde{k}^2 \Pi = 0, \quad \mathbf{x}' = (x_1, x_2) \in \Omega \tag{88}$$

$$\Delta \Psi + \tilde{k}^2 \Psi = 0, \quad \tilde{k}^2 = \varepsilon - \gamma^2, \tag{89}$$

satisfying the boundary conditions on Γ_0

$$\Pi|_{\Gamma_0} = 0, \quad \left.\frac{\partial \Psi}{\partial n}\right|_{\Gamma_0} = 0, \tag{90}$$

In fact, it is necessary to determine only one function, H_3 for the TE-polarization or E_3 for the TM-polarization; the remaining components are obtained using differentiation.

LECTURE 1

STATEMENTS AND ANALYSIS OF THE BVPS FOR MAXWELLS AND HELMHOLTZ EQUATIONS

In the two-dimensional case, the Helmholtz equation $\mathcal{L}(k^2)u = 0$ written in the polar coordinates $\mathbf{r} = (r, \phi)$ has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + k^2 u = 0. \tag{91}$$

Assume that the function $u = u(\mathbf{r})$ satisfies the Helmholtz equation outside a circle of radius r_0. On any circle of radius $r > r_0$ function u can be decomposed in a trigonometric Fourier series

$$u(\mathbf{r}) = \sum_{n=-\infty}^{\infty} u_n(r)e^{in\phi} \quad (0 < \phi < 2\pi), \tag{92}$$

where the coefficients

$$u_n(r) = \frac{1}{2\pi}\int_0^{2\pi} u(\mathbf{r})e^{-in\phi}d\phi \tag{93}$$

are functions of r. In order to find $u_n(r)$ multiply equations (91) by $\frac{1}{2\pi}e^{-in\phi}$ and integrate over a circle of radius r.

As a result of integration, we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_n}{dr}\right) - \frac{n^2}{r^2}u_n + k^2 u_n = 0, \quad n = 0, \pm 1, \ldots. \tag{94}$$

(94) is a second-order ordinary differential equation with constant coefficients for $u_n(r)$ which holds for $r > r_0$. Equation (94) is actually *the Bessel equation of order n*. Its general solution can be written as

$$u_n(r) = A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr), \tag{95}$$

where $H_n^{(1,2)}(z)$ are its linearly independent solutions; they are the nth-order Hankel functions of the first and second kind, respectively.

Thus any solution $u = u(\mathbf{r})$ to the homogeneous Helmholtz equation (satisfied outside a circle of radius r_0) can be represented for $r > r_0$ in the form of a series

$$u(\mathbf{r}) = \sum_{n=-\infty}^{\infty} [A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr)]e^{in\phi} \quad (0 < \phi < 2\pi, \ r > r_0). \tag{96}$$

At infinity, the following asymptotical formulas are valid

$$H_n^{(1,2)}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right)} + O\left(\frac{1}{z^{3/2}}\right), \tag{97}$$

which yields an asymptotic estimate of the solution to the homogeneous Helmholtz equation at infinity

$$u(\mathbf{r}) = O\left(\frac{1}{\sqrt{r}}\right). \tag{98}$$

For the zero-order Hankel functions of the first and second kind, respectively, the following asymptotical formulas are valid

$$
\begin{aligned}
H_0^{(1)}(z) &= \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{\pi}{4}\right)} + \dots, \\
H_0^{(2)}(z) &= \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{\pi}{4}\right)} + \dots,
\end{aligned} \tag{99}
$$

LECTURE 1: STATEMENTS AND ANALYSIS OF THE BVPS FOR MAXWELLS AND HELMHOLTZ EQUATIONS.

Let us recall first that the plane waves propagation along the x-axis have the form

$$\hat{u} = f\left(t - \frac{x}{a}\right), \qquad \hat{\hat{u}} = f\left(t + \frac{x}{a}\right), \tag{100}$$

where $\hat{u}$ and $\hat{\hat{u}}$ are, respectively, the forward wave (propagating in the positive direction of the x-axis) and backward wave (propagating in the negative direction of the x-axis). They satisfy the following first-order partial differential equations

$$\frac{\partial \hat{u}}{\partial x} + \frac{1}{a}\frac{\partial \hat{u}}{\partial t} = 0, \tag{101}$$

$$\frac{\partial \hat{\hat{u}}}{\partial x} - \frac{1}{a}\frac{\partial \hat{\hat{u}}}{\partial t} = 0. \tag{102}$$

In the stationary mode

$$u = v(x)e^{i\omega t} \tag{103}$$

For the amplitude function v these relations take the form

$$\frac{\partial \hat{v}}{\partial x} + ik\hat{v} = 0, \tag{104}$$

$$\frac{\partial \hat{\hat{v}}}{\partial x} - ik\hat{\hat{v}} = 0, \tag{105}$$

for the forward and backward waves, respectively, where $k = \dfrac{\omega}{a}$.

Spherical waves. If a spherical wave is excited by the sources situated in a bounded part of the space (not at infinity), then at large distances from the source, a spherical wave is similar to a plane wave whose amplitude decays as $\dfrac{1}{r}$. This natural physical assumption leads to a conclusion that the outgoing, respectively, incoming, spherical waves must satisfy the relationships

$$\frac{\partial u}{\partial r} + \frac{1}{a}\frac{\partial u}{\partial t} = o\left(\frac{1}{r}\right), \tag{106}$$

$$\frac{\partial u}{\partial r} - \frac{1}{a}\frac{\partial u}{\partial t} = o\left(\frac{1}{r}\right). \tag{107}$$

For the amplitude functions in the stationary mode we have

$$\frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right) \quad \text{for outgoing spherical waves,} \tag{108}$$

$$\frac{\partial v}{\partial r} - ikv = o\left(\frac{1}{r}\right) \quad \text{for incoming spherical waves.} \tag{109}$$

Let us prove now that at large distances from the source, any outgoing spherical wave decays as $\frac{1}{r}$.

1. In the case of a point source at the origin, this statement is trivial because the wave itself has the form

$$u(r,t) = \frac{e^{i(\omega t - kr)}}{r} = v_0(r)e^{i\omega t}, \tag{110}$$

so that

$$\frac{\partial v_0}{\partial r} + ikv_0 = o\left(\frac{1}{r}\right). \tag{111}$$

Check this relationship.

LECTURE 1: STATEMENTS AND ANALYSIS OF THE BVPS FOR MAXWELLS AND HELMHOLTZ EQUATIONS.

2. Let a spherical wave be excited by a point source situated at a point $\mathbf{r}_0$. The amplitude of the spherical wave is

$$v_0(\mathbf{r}) = \frac{e^{ikR}}{R}, \quad R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta}. \tag{112}$$

Calculating the derivative we obtain

$$\frac{\partial R}{\partial r} = \frac{r - r_0\cos\theta}{R} \sim 1 + O\left(\frac{1}{r}\right) \tag{113}$$

and

$$\frac{\partial v_0}{\partial R} + ikv_0 = o\left(\frac{1}{R}\right).$$

in view of (111). Next,

$$\frac{\partial v_0}{\partial r} = \frac{\partial v_0}{\partial R}\frac{\partial R}{\partial r} = \frac{\partial v_0}{\partial R}\left(1 + O\left(\frac{1}{r}\right)\right) = \frac{\partial v_0}{\partial R} + o\left(\frac{1}{r}\right)$$

because

$$\frac{\partial v_0}{\partial R} \cdot O\left(\frac{1}{r}\right) = o\left(\frac{1}{r}\right).$$

Finally,

$$\frac{\partial v_0}{\partial r} + ikv_0 + o\left(\frac{1}{r}\right) = o\left(\frac{1}{r}\right) \tag{114}$$

what is to be proved.

3. Show that the volume potential

$$v(\mathbf{r}) = \int_T f(\mathbf{r}_0) \frac{e^{-ikR}}{R} d\tau_{\mathbf{r}_0}, \quad R = |\mathbf{r} - \mathbf{r}_0|, \tag{115}$$

satisfies condition (108). Introducing the notation

$$\mathcal{P}v = \frac{\partial v}{\partial r} + ikv, \tag{116}$$

we obtain

$$\mathcal{P}v = \int_T f(\mathbf{r}_0) \mathcal{P}\left(\frac{e^{-ikR}}{R}\right) d\tau_{\mathbf{r}_0} = \int_T f(\mathbf{r}_0) o\left(\frac{1}{r}\right) d\tau_{\mathbf{r}_0} = o\left(\frac{1}{r}\right). \tag{117}$$

Volume potential (115) is the amplitude of an outgoing wave excited by the sources distributed arbitrarily in a bounded domain T. Also, function v defined by (115) satisfies the inhomogeneous Helmholtz equation $\mathcal{L}(k^2)u = -f$ and decays as $\frac{1}{r}$ for $r \to \infty$. In addition, it satisfies the condition

$$\frac{\partial v}{\partial r} + ikv = o\left(\frac{1}{r}\right). \tag{118}$$

Theorem 20

There is one and only one solution to the inhomogeneous Helmholtz equation

$$\mathcal{L}(k^2)v = (\Delta + k^2)v = -f(\mathbf{r}), \tag{119}$$

where $f(\mathbf{r})$ is a function with local support, which satisfies the conditions at infinity

$$
\begin{aligned}
v &= O\left(\frac{1}{r}\right), \\
\frac{\partial v}{\partial r} + ikv &= o\left(\frac{1}{r}\right).
\end{aligned}
\tag{120}
$$

Proof. Assuming that there are two different solutions v_1 and v_2 and setting

$$w = v_1 - v_2,$$

we see that w satisfies the homogeneous Helmholtz equation $\mathcal{L}(k^2)w = 0$ and the conditions at infinity (120). Let Σ_R be a sphere of radius R (later, we will take the limit $R \to \infty$). Applying the third Green formula to $w(\mathbf{r})$ and the fundamental solution $\phi_0(\mathbf{r}_0) = \dfrac{e^{-ikR}}{4\pi R}$, $R = |\mathbf{r}_0 - \mathbf{r}|$, we arrive at the integral representation of w at a point $\mathbf{r} \in \Sigma_R$

$$w(\mathbf{r}) = \int_{\Sigma_R} \left[\phi_0(\mathbf{r}_0)\frac{\partial w}{\partial r} - w\frac{\partial}{\partial r}\left(\phi_0(\mathbf{r}_0)\right) \right] d\sigma_{\mathbf{r}_0}. \tag{121}$$

The conditions at infinity (120) for $w(\mathbf{r})$ and $\phi_0(\mathbf{r})$ yield

$$
\begin{aligned}
\phi_0 \frac{\partial w}{\partial r} - w \frac{\partial}{\partial \nu}(\phi_0) &= \phi_0 \left[-ikw + o\left(\frac{1}{r}\right) \right] - \\
&- w \left[-ik\phi_0 + o\left(\frac{1}{r}\right) \right] = \phi_0 o\left(\frac{1}{r}\right) - w o\left(\frac{1}{r}\right) = o\left(\frac{1}{r^2}\right).
\end{aligned}
\tag{122}
$$

Therefore,

$$
w(\mathbf{r}) = \int_{\Sigma_R} o\left(\frac{1}{r^2}\right) d\sigma_{\mathbf{r}_0} \to 0, \quad R \to \infty.
\tag{123}
$$

This implies $w(\mathbf{r}) = 0$ at any $\mathbf{r} \in \Sigma_R$ and thus at any spatial $\mathbf{r}$.
Conditions (120) are called *Sommerfeld radiation conditions*.
In the two-dimensional case the Sommerfeld radiation conditions at infinity take the form

$$
\begin{aligned}
v &= O\left(\frac{1}{\sqrt{r}}\right), \\
\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial v}{\partial r} + ikv\right) &= 0.
\end{aligned}
\tag{124}
$$

Theorem 21

Let $u_0(\mathbf{r})$ be a solution to the Helmholtz equation satisfied outside a circle of radius r_0. If

$$\lim_{r \to \infty} \int_{C_r} |u|^2 dl = 0, \tag{125}$$

where C_r is a circle of radius r, then $u \equiv 0$ for $r > r_0$.

Proof. Any solution $u = u(\mathbf{r})$ to the (homogeneous) Helmholtz equation (satisfied outside a circle of radius r_0) can be represented for $r > r_0$ in the form of series (96)

$$u(\mathbf{r}) = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\phi}, \quad u_n(r) = A_n H_n^{(1)}(kr) + B_n H_n^{(2)}(kr) \quad (0 < \phi < 2\pi, \ r > r_0). \tag{126}$$

Therefore,

$$\lim_{r \to \infty} \int_{C_r} |u|^2 dl = 2\pi \sum_{n=-\infty}^{\infty} r|u_n(r)|^2. \tag{127}$$

If

$$\lim_{r \to \infty} \int_{C_r} |u|^2 dl = 0,$$

then (127) yields

$$\lim_{r \to \infty} r|u_n(r)|^2 = 0, \quad n = 0, \pm 1, \pm 2, \dots. \tag{128}$$

Next, according to asymptotical formulas (97) for Hankel functions $r|u_n(r)|^2$ are bounded quantities at $r \to \infty$, namely,

$$r|u_n(r)|^2 = rO\left(\frac{1}{r}\right) = O(1), \quad n = 0, \pm 1, \pm 2, \dots, \tag{129}$$

which, together with (128), implies

$$A_n = B_n = 0, \quad n = 0, \pm 1, \pm 2, \dots, \tag{130}$$

and, consequently, $u \equiv 0$ for $r > r_0$ in line with representation (126).

Theorem 22

Let $u_0(\mathbf{r})$ be a solution to the Helmholtz equation satisfied outside a sphere S_{r_0} of radius r_0. If

$$\lim_{r \to \infty} \int_{S_r} |u|^2 ds = 0, \tag{131}$$

then $u \equiv 0$ for $r > r_0$.

For the vector solutions of Maxwell equations (69) and (70), electromagnetic field $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$, the similar statements are valid

Theorem 23

Let $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ be a solution to the Maxwell equation system satisfied outside a sphere of radius r_0. If

$$\lim_{r \to \infty} \int_{S_r} |[\mathbf{H}, \mathbf{e}_r]|^2 ds = 0, \tag{132}$$

or

$$\lim_{r \to \infty} \int_{S_r} |[\mathbf{E}, \mathbf{e}_r]|^2 ds = 0, \tag{133}$$

where S_r is a sphere of radius r and $\mathbf{e}_r = \mathbf{r}/r$ is the unit position vector of the points on S_r, then $\mathbf{E}(\mathbf{r}) \equiv 0$, $\mathbf{H}(\mathbf{r}) \equiv 0$ for $r > r_0$.

Formulate a **scalar (acoustical) problem of the wave diffraction by a transparent body** Ω_1. Let Ω_1 be a domain bounded by a piecewise smooth surface Σ. The problem under consideration is reduced to a BVPs for the inhomogeneous Helmholtz equation with a piecewise constant coefficient

$$
\begin{array}{rcll}
\Delta u_0(\mathbf{r}) + k_0^2 u_0(\mathbf{r}) & = & -f_0, & \mathbf{r} \in \Omega_0 = R^3 \setminus \overline{\Omega_1}, \\
\Delta u_1(\mathbf{r}) + k_1^2 u_1(\mathbf{r}) & = & -f_1, & \mathbf{r} \in \Omega_1;
\end{array}
\tag{134}
$$

solution u satisfies the conjugation conditions on Σ

$$
u_1 - u_0 = 0, \qquad \frac{\partial u_1}{\partial n} - \frac{\partial u_0}{\partial n} = 0,
\tag{135}
$$

and the conditions at infinity

$$
\begin{array}{rcl}
u_0 & = & O\left(\dfrac{1}{r}\right), \\[2mm]
\dfrac{\partial u_0}{\partial r} - i k_0 u_0 & = & o\left(\dfrac{1}{r}\right).
\end{array}
\tag{136}
$$

Theorem 24

The solution to problem (134)–(136) is unique.

Proof. Since problem (134)–(136) is linear, it is sufficient to prove that the corresponding homogeneous problem (with $f_0 = f_1 = 0$) has only a trivial solution. Together with u_0 and u_1 consider the corresponding complex conjugate functions u_0^* and u_1^*. They satisfy the same boundary and transmission conditions; however, the condition at infinity takes the form

$$\frac{\partial u_0^*}{\partial r} + ik_0 u_0^* = o\left(\frac{1}{r}\right). \tag{137}$$

Applying the second Green formula to u_1^* and u_1^* in domain Ω_1, we obtain

$$\int_\Sigma \left[u_1 \frac{\partial u_1^*}{\partial \nu} - u_1^* \frac{\partial u_1}{\partial \nu} \right] d\sigma_{\mathbf{r}_0} = 0, \tag{138}$$

where ν denotes the unit normal vector to the boundary Σ directed into the exterior of Ω_1. Let S_R be a sphere of sufficiently large radius R containing domain Ω_1. Applying the second Green formula to u_0 and u_0^* in the domain Ω_S situated between Ω_1 and S_R, we obtain

$$\int_\Sigma \left[u_0 \frac{\partial u_0^*}{\partial \nu_0} - u_0^* \frac{\partial u_0}{\partial \nu_0} \right] d\sigma_{\mathbf{r}_0} + \int_{S_R} \left[u_0 \frac{\partial u_0^*}{\partial r} - u_0^* \frac{\partial u_0}{\partial r} \right] d\sigma_{\mathbf{r}_0} = 0, \tag{139}$$

where $\partial \partial \nu_0$ denotes the directional derivative in the direction of the unit normal vector ν to Σ directed into the interior of Ω_1 (external with respect to Ω_0).

Adding up (138) and (139) and taking into account the conjugation conditions on Σ, we have

$$\int_{S_R} \left[u_0 \frac{\partial u_0^*}{\partial r} - u_0^* \frac{\partial u_0}{\partial r} \right] d\sigma_{\mathbf{r}_0} = 0. \tag{140}$$

Applying the condition at infinity and transferring to the limit $R \to \infty$ in (140) we obtain

$$\lim_{R \to \infty} \int_{S_R} |u_0|^2 ds = 0, \tag{141}$$

Thus $u_0 \equiv 0$ outside sphere S_R according to Theorem 22. Applying the third Green formula (121) in Ω_S we obtain that $u_0 \equiv 0$ in Ω_S. Then applying the third Green formula in Ω_1 we obtain that $u_1 \equiv 0$ in Ω_1. Therefore, homogeneous problem (134)–(136) has only a trivial solution. The theorem is proved.

Formulate a **vector (electromagnetic) problem of the wave diffraction by a transparent body** Ω_1. Let Ω_1 be a domain bounded by a piecewise smooth surface Σ and $\Omega_0 = R^3 \setminus \overline{\Omega_1}$. The problem under consideration is reduced to a BVP for the inhomogeneous system of Maxwell equations (69) and (70) with a piecewise constant coefficient

$$\operatorname{rot} \mathbf{H_j} = -i\omega\epsilon_j \mathbf{E_j} + \mathbf{J_j}, \quad \operatorname{rot} \mathbf{E_j} = i\omega\mu_j \mathbf{H_j}, \quad j = 0,\, 1, \tag{142}$$

with the transmission conditions stating the continuity of the tangential field components across interface Σ

$$[\mathbf{H}_1, \nu] = [\mathbf{H}_0, \nu], \quad [\mathbf{E}_1, \nu] = [\mathbf{E}_0, \nu], \tag{143}$$

and the Silver–Müller radiation conditions at infinity

$$\lim_{r \to \infty} r\left([\mathbf{H}_0, \mathbf{e}_r] - ik_0 \mathbf{E}_0\right) = 0, \quad k_0 = \omega\sqrt{\epsilon_0\mu_0}, \tag{144}$$

where ν is the unit normal vector to Σ, $\mathbf{e}_r = \mathbf{r}/r$ is the unit position vector of the points on S_r and the limit holds uniformly with respect to all directions (specified by $\mathbf{e}_r$). Note that in this case (142) can be written equivalently (in every domain where the parameters are constant) as a one vector equation with respect to i.e. $\mathbf{E}(\mathbf{r})$ by eliminating $\mathbf{H}(\mathbf{r})$:

$$\operatorname{rot}\operatorname{rot} \mathbf{E_j} - \omega^2 \epsilon_j \mu_j \mathbf{E_j} = \tilde{\mathbf{J}}_\mathbf{j}, \quad j = 0,\, 1. \tag{145}$$

Theorem 25

The solution to problem (142)–(144) is unique.

Proof. Since problem (142)–(144) is linear, it is sufficient to prove that the corresponding homogeneous problem (with $\mathbf{J_j} = 0$) has only a trivial solution. Next, one has to apply Theorem 23 and perform the same steps as in the proof of Theorem 24 using Lorentz lemma instead of the Green formulas.
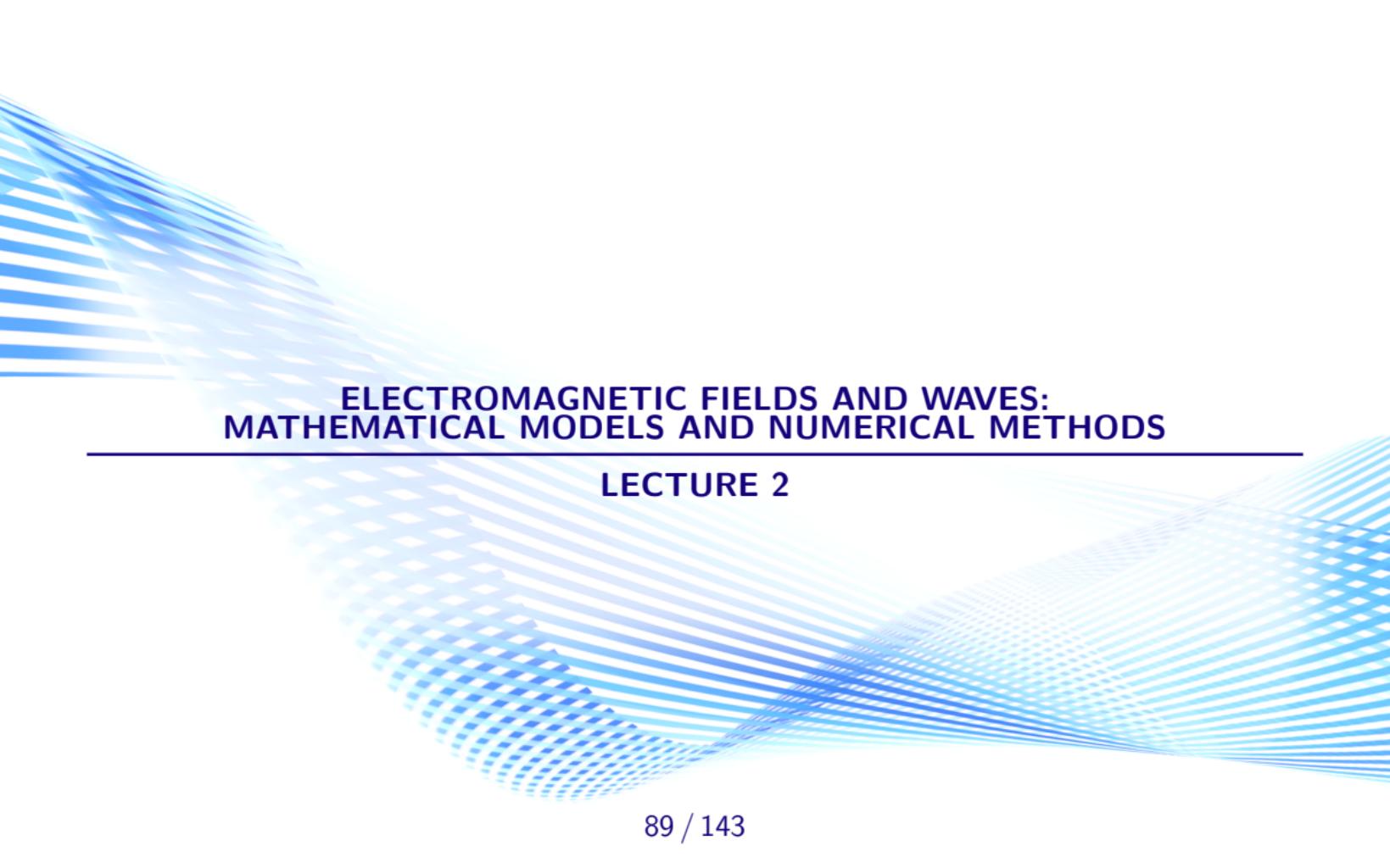
ELECTROMAGNETIC FIELDS AND WAVES:
MATHEMATICAL MODELS AND NUMERICAL METHODS

LECTURE 2
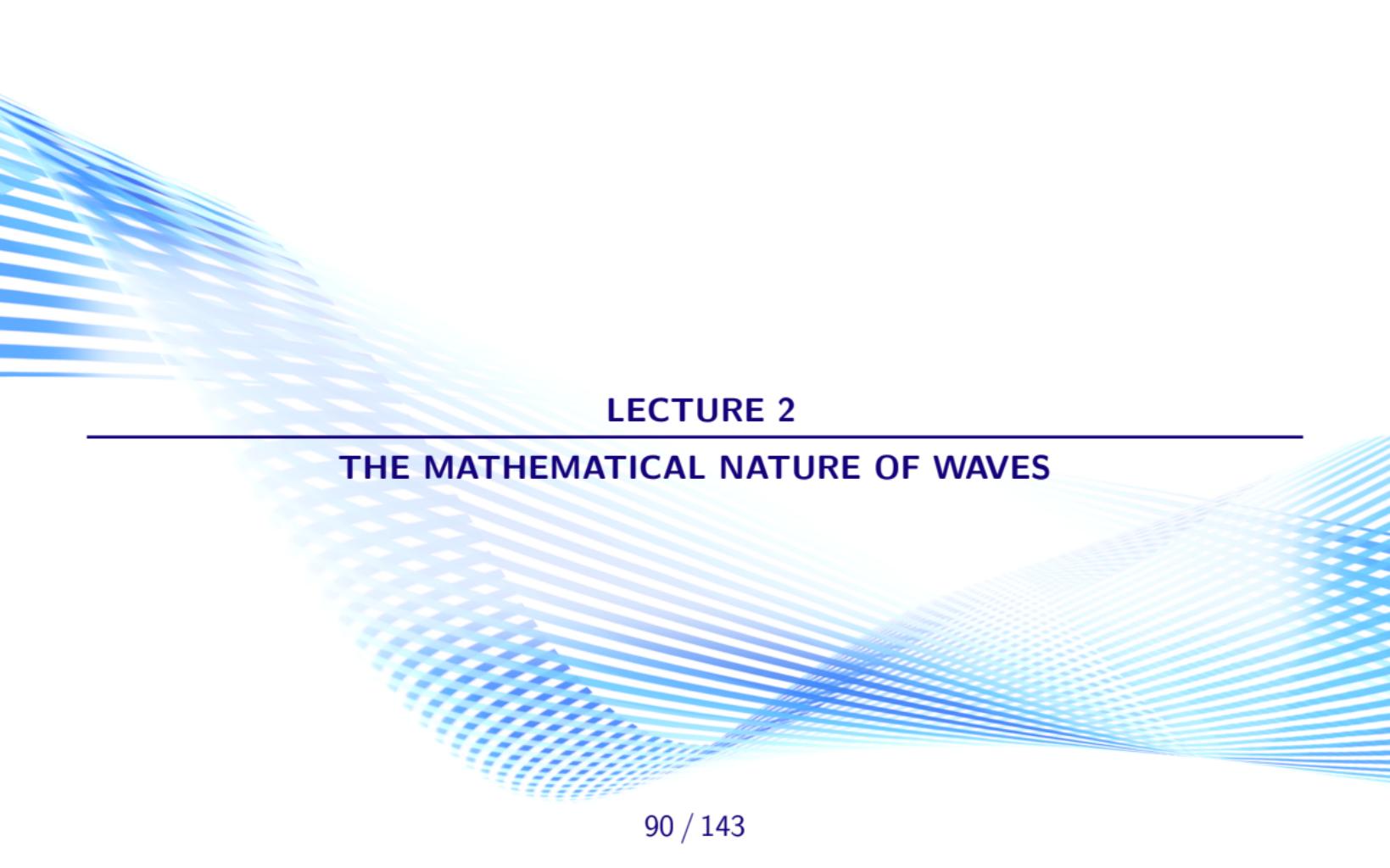
LECTURE 2

THE MATHEMATICAL NATURE OF WAVES

Going back to the problems on normal waves we see that the form of solution in (85)

$$
\begin{aligned}
\mathbf{E}(\mathbf{x}) &= \left(E_1(\mathbf{x}')\,\mathbf{e}_1 + E_2(\mathbf{x}')\,\mathbf{e}_2 + E_3(\mathbf{x}')\,\mathbf{e}_3\right)e^{i\gamma x_3}, \\
\mathbf{H}(\mathbf{x}) &= \left(H_1(\mathbf{x}')\,\mathbf{e}_1 + H_2(\mathbf{x}')\,\mathbf{e}_2 + H_3(\mathbf{x}')\,\mathbf{e}_3\right)e^{i\gamma x_3}, \\
\mathbf{x}' &= (x_1,\ x_2),
\end{aligned}
\tag{146}
$$

with the dependence $e^{i\gamma x_3}$ on longitudinal coordinate x_3 specify a wave propagating in the positive direction of x_3-axis. Problems on normal waves (88)–(90) have nontrivial solutions if

$$
\tilde{k}^2 = \varepsilon - \gamma^2 = \lambda_n^D \quad \text{or} \quad \tilde{k}^2 = \lambda_n^N, \quad n = 1, 2, \ldots,
\tag{147}
$$

so that the eigenvalues of normal waves

$$
\gamma = \gamma_n^D = \sqrt{\varepsilon - \lambda_n^D} \quad \text{or} \quad \gamma = \gamma_n^N = \sqrt{\varepsilon - \lambda_n^N}.
\tag{148}
$$

We have $0 \le \lambda_1^{D,N} \le \lambda_2^{D,N} \le \ldots$; therefore, that are at most finitely many values of γ_n^D and γ_n^N that are real, while infinitely many of them are purely imaginary. Consequently, according to (146), there are at most finitely many normal waves that propagate without attenuation (in the positive direction of x_3-axis) and infinitely many decay exponentially.

LECTURE 2: THE MATHEMATICAL NATURE OF WAVES.

Propagation of electromagnetic waves along the waveguide is described by the homogeneous system of Maxwell equations which can be written in the form

$$\begin{aligned} \text{rot}\,\mathbf{H} &= -ik\mathbf{E}, \\ \text{rot}\,\mathbf{E} &= ik\mathbf{H}, \end{aligned} \tag{149}$$

with the boundary conditions for the tangential electric field components on the perfectly conducting walls Σ of the waveguide

$$\mathbf{E}_\tau|_\Sigma = 0, \tag{150}$$

Look for particular solutions of (149) in the form

$$\begin{aligned} \mathbf{E} &= \text{grad}\,\text{div}\,\mathbf{P} + k^2\mathbf{P}, \\ \mathbf{H} &= -ik\text{rot}\,\mathbf{P}, \end{aligned} \tag{151}$$

using the polarization potential $\mathbf{P} = [0, 0, \Pi]$ that has only one nonzero component $P_3 = \Pi$. It is easy to see that

$$H_3 = 0, \quad \mathbf{E} = [0, 0, E_3], \quad \mathbf{H} = [H_1, H_2, 0], \tag{152}$$

and this case is called TM-polarization or E-polarization Substituting (151) into (149) yields the equations

$$\begin{aligned} \Delta_3\Pi + k^2\Pi &= 0 \quad \text{or} \quad \Delta\Pi + \frac{\partial^2\Pi}{\partial x_3^2} + k^2\Pi = 0, \\ \Delta_3 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \end{aligned} \tag{153}$$

Condition (150) is satisfied if we assume that

$$\Pi|_{\Sigma} = 0, \tag{154}$$

because the third components of both $\mathbf{P}$ and $\mathbf{E}$ are actually tangential components that must vanish on the waveguide wall and they are coupled by the first relation (151). (153) and (154) constitute the Dirichlet BVP for the Helmholtz equation in the tube. We look for the solution to this problem in the form

$$\Pi(\mathbf{x}) = \Pi(\mathbf{x}', x_3) = \psi(\mathbf{x}')f(x_3), \quad \mathbf{x}' = (x_1, x_2), \quad \psi(\mathbf{x}'),\ f(x_3) \neq 0, \tag{155}$$

using the separation of variables. Namely, substituting (155) into (153) and dividing by nonvanishing product $f\psi$ we have

$$f\Delta\psi + f''\psi + k^2 f\psi = 0 \quad \text{or} \quad \frac{\Delta\psi}{\psi} + \frac{f''}{f} = -k^2, \tag{156}$$

which yields

$$\frac{\Delta\psi}{\psi} = -\lambda, \quad \frac{f''}{f} = \lambda - k^2 \tag{157}$$

with a certain constant λ.

Thus ψ must solve the Dirichlet eigenvalue problem for the Laplace equation in cross-sectional domain Ω

$$\begin{aligned} \Delta\psi + \lambda\psi &= 0, \quad \mathbf{x}' \in \Omega, \\ \psi|_\Gamma &= 0. \end{aligned} \tag{158}$$

Denote by $\Lambda = \{\lambda_n\}$ and $\Psi = \{\psi_n\}$ the system of eigenvalues and eigenfunctions of this problem. A particular solution of (153) is

$$\Pi = \Pi_n(\mathbf{x}) = \psi_n(\mathbf{x}')f_n(x_3), \tag{159}$$

where f_n satisfies the equation

$$f_n'' + (k^2 - \lambda_n)f_n = 0. \tag{160}$$

The general solution of (160) is

$$f_n(x_3) = A_n e^{i\gamma_n x_3} + B_n e^{-i\gamma_n x_3}, \quad \gamma_n = \sqrt{k^2 - \lambda_n}. \tag{161}$$

The first and the second terms in (161) correspond, respectively, to the wave propagating in the positive or negative direction of the waveguide axis.

Considering the wave propagating in the positive direction set

$$f_n(x_3) = A_n e^{i\gamma_n x_3}. \tag{162}$$

As a result we obtain the solution

$$\Pi_n(\mathbf{x}', x_3) = A_n \psi_n(\mathbf{x}') e^{i\gamma_n x_3}. \tag{163}$$

We have $0 \le \lambda_1 \le \lambda_2 \le \dots$; therefore, that are at most finitely many values of $\gamma_n = \sqrt{k^2 - \lambda_n}$ with $k^2 > \lambda_n$ that are real, while infinitely many of them, for $\gamma_n = i\sqrt{\lambda_n - k^2}$ ($i^2 = -1$) with $k^2 < \lambda_n$, are purely imaginary. Consequently, there are at most finitely many waves in the waveguide that propagate without attenuation (in the positive direction of x_3-axis) and infinitely many decay exponentially.

Looking for particular solutions of (149) in the form

$$
\begin{aligned}
\mathbf{H} &= \operatorname{grad} \operatorname{div} \mathbf{P} + k^2 \mathbf{P}, \\
\mathbf{E} &= ik \operatorname{rot} \mathbf{P},
\end{aligned}
\tag{164}
$$

where the polarization potential $\mathbf{P} = [0, 0, \Pi]$ has only one nonzero component $P_3 = \Pi$, it is easy to see that

$$
E_3 = 0, \quad \mathbf{H} = [0, 0, H_3], \quad \mathbf{E} = [E_1, E_2, 0],
\tag{165}
$$

and this case is called TE-polarization or H-polarization. Substituting (164) into (149) yields the equations

$$
\begin{aligned}
\Delta_3 \Pi + k^2 \Pi &= 0 \quad \text{or} \quad \Delta \Pi + \frac{\partial^2 \Pi}{\partial x_3^2} + k^2 \Pi = 0, \\
\Delta_3 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.
\end{aligned}
\tag{166}
$$

Condition (150) is satisfied if we assume that

$$
\left. \frac{\partial \Pi}{\partial n} \right|_\Sigma = 0,
\tag{167}
$$

because the third components of $\mathbf{P}$ and the first two of $\mathbf{E}$ are tangential components that must vanish on the waveguide wall and they are coupled by the first relation (164).

Repeating the above analysis we see that

$$\Pi = \Pi_n(\mathbf{x}) = A_n \psi_n(\mathbf{x}') e^{i\gamma_n x_3}, \tag{168}$$

where ψ_n solves the Neumann eigenvalue problem for the Laplace equation in cross-sectional domain Ω

$$\begin{aligned}
\Delta\psi + \lambda\psi &= 0, \quad \mathbf{x}' \in \Omega, \\
\left.\frac{\partial\psi}{\partial n}\right|_{\Sigma} &= 0.
\end{aligned} \tag{169}$$

(168) specifies the wave propagating in the positive direction of the waveguide axis. Denote by $\Lambda^H = \{\lambda_n^H\}$ and $\Psi^H = \{\psi_n^H\}$ the system of eigenvalues and eigenfunctions of this problem. We have $0 \leq \lambda_1^H \leq \lambda_2^H \leq \dots$; therefore, that are at most finitely many values of $\gamma_n^H = \sqrt{k^2 - \lambda_n^H}$ with $k^2 > \lambda_n^H$ that are real, while infinitely many of them, for $\gamma_n^H = i\sqrt{\lambda_n^H - k^2}$ with $k^2 < \lambda_n^H$ are purely imaginary. Consequently, there are at most finitely many waves in the waveguide that propagate without attenuation (in the positive direction of x_3-axis) and infinitely many decay exponentially.
The waves obtained from (151), (152) or (164), (165) are called, respectively, TM-waves or TE-waves.

Diffraction from a dielectric obstacle in a 2D-guide. Introduce the complex magnitude of the stationary electric and magnetic field, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$, respectively, where $\mathbf{r} = (x, y, z)$, and consider the problem of diffraction of a TM wave (or mode)

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) \exp(-i\omega t), \quad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r}) \exp(-i\omega t), \tag{170}$$

$$\mathbf{E}(\mathbf{r}) = (E_x, 0, 0), \quad \mathbf{H}(\mathbf{r}) = \left(0, \frac{1}{i\omega\mu_0}\frac{\partial E_x}{\partial z}, -\frac{1}{i\omega\mu_0}\frac{\partial E_x}{\partial y}\right), \tag{171}$$

by a dielectric inclusion D in a parallel-plane waveguide $\mathcal{W} = \{\mathbf{r} : 0 < y < \pi, -\infty < x, z < \infty\}$.
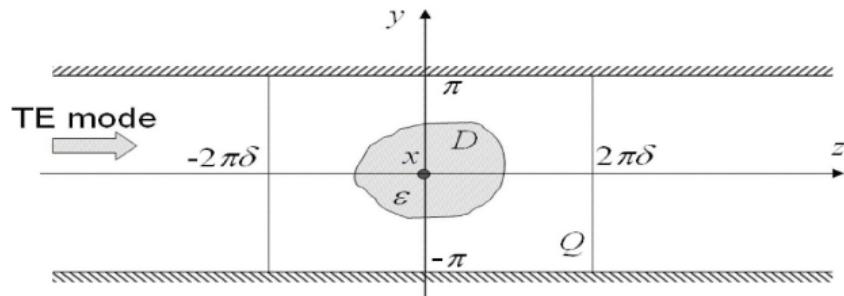

Figure .1: TE-mode diffraction by a dielectric inclusion in a parallel-plane waveguide

The total field $u(y, z) = E_x(y, z) = E_x^{inc}(y, z) + E_x^{scat}(y, z) = u^i(y, z) + u^s(y, z)$ of the diffraction by the D of the unit-magnitude TE wave with the only nonzero component is the solution to the BVP

$$[\Delta + \kappa^2 \varepsilon(y, z)]u(y, z) = 0 \text{ in } S = \{(y, z) : 0 < y < \pi, -\infty < z < \infty\}, \quad u(\pm\pi, z) = 0, \tag{172}$$

$$u(y, z) = u^i(y, z) + u^s(y, z), \quad u^s(y, z) = \sum_{n=1}^{\infty} a_n^{\pm} \exp(i\Gamma_n z)\sin(ny), \tag{173}$$

where $\Delta = \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is the Laplace operator, superscripts $^+$ and $^-$ correspond, respectively, to the domains $z > 2\pi\delta$ and $z < -2\pi\delta$, $\omega = \kappa c$ is the dimensionless circular frequency, $\kappa = \omega/c = 2\pi/\lambda$ is the dimensionless frequency parameter (λ is the free-space wavelength), $c = (\varepsilon_0 \mu_0)^{-1/2}$ is the speed of light in vacuum, and $\Gamma_n = (\kappa^2 - n^2)^{1/2}$ is the transverse wavenumber satisfying the conditions

$$Im\,\Gamma_n \geq 0, \quad \Gamma_n = i|\Gamma_n|, \quad |\Gamma_n| = Im\,\Gamma_n = (n^2 - \kappa^2)^{1/2}, \quad n > \kappa. \tag{174}$$

It is also assumed that the series in (173) converges absolutely and uniformly and allows for double differentiation with respect to y and z.
Note that $u^i(y, z)$ satisfies (172) in S, the boundary condition, and radiation condition (173) only in the positive direction, so that the electromagnetic field with the x-component $u^i(y, z)$ may be interpreted as a normal wave (a waveguide mode) coming from the domain $z < -2\pi\delta$.

Diffraction from a dielectric obstacle in a 3D-guide. Diffraction of electromagnetic waves by a dielectric body Q in a 3D tube (a waveguide) with cross section Ω (a 2D domain bounded by smooth curve Γ) parallel to the x_3-axis in the cartesian coordinate system is described by the solution to the inhomogeneous system of Maxwell equations

$$\begin{aligned}
\mathbf{rot}\mathbf{H} &= -i\omega\hat{\epsilon}\mathbf{E} + \mathbf{j}_E^0 \\
\mathbf{rot}\mathbf{E} &= i\omega\mu_0\mathbf{H},
\end{aligned} \tag{175}$$

$$\mathbf{E}_\tau|_{\partial P} = 0, \quad \mathbf{H}_\nu|_{\partial P} = 0, \tag{176}$$

admitting for $|x_3| > C$ and sufficiently large $C > 0$ the representations ($+$ corresponds to $+\infty$ and $-$ to $-\infty$)

$$\begin{aligned}
\binom{\mathbf{E}}{\mathbf{H}} &= \sum_p R_p^{(\pm)} e^{-i\gamma_p^{(1)}|x_3|} \binom{\lambda_p^{(1)}\Pi_p e_3 - i\gamma_p^{(1)}\nabla_2\Pi_p}{-i\omega\varepsilon_0(\nabla_2\Pi_p)\times e_3} + \\
&+ \sum_p Q_p^{(\pm)} e^{-i\gamma_p^{(2)}|x_3|} \binom{i\omega\mu_0(\nabla_2\Psi_p)\times e_3}{\lambda_p^{(2)}\Psi_p e_3 - i\gamma_p^{(2)}\nabla_2\Psi_p}.
\end{aligned} \tag{177}$$

Here, $\gamma_p^{(j)} = \sqrt{k_0^2 - \lambda_p^{(j)}}$, $\operatorname{Im} \gamma_p^{(j)} < 0$ or $\operatorname{Im} \gamma_p^{(j)} = 0$, $k_0 \gamma_p^{(j)} \geq 0$, and $\lambda_p^{(1)}$, $\Pi_p(x_1, x_2)$ and $\lambda_p^{(2)}$, $\Psi_p(x_1, x_2)$ ($k_0^2 = \omega^2 \varepsilon_0 \mu_0$) are the complete system of eigenvalues and orthogonal and normalized in $L_2(\Pi)$ eigenfunctions of the two-dimensional Laplace operator $-\Delta$ in the rectangle $\Pi_{ab} = \{(x_1, x_2) : 0 < x_1 < a, 0 < x_2 < b\}$ with the Dirichlet and the Neumann conditions, respectively; and $\nabla_2 \equiv e_1 \, \partial/\partial x_1 + e_2 \, \partial/\partial x_2$.

We assume that $\mathbf{E}^0$ and $\mathbf{H}^0$ are solutions of BVP under consideration in the absence of body Q, $\hat{\epsilon}(x) = \epsilon_0 \hat{I}$, $x \in P$ ($\hat{I}$ is the identity tensor):

$$\begin{aligned}
\operatorname{rot}\mathbf{H}^0 &= -i\omega\epsilon_0\mathbf{E}^0 + \mathbf{j}_E^0 \\
\operatorname{rot}\mathbf{E}^0 &= i\omega\mu_0\mathbf{H}^0,
\end{aligned} \tag{178}$$

$$\mathbf{E}_\tau^0|_{\partial P} = 0, \quad \mathbf{H}_\nu^0|_{\partial P} = 0. \tag{179}$$

These solutions can be expressed in an analytical form in terms of $\mathbf{j_E^0}$ using Green's tensor of domain P. These solutions should not satisfy the conditions at infinity (177). For example, $\mathbf{E}^0$ and $\mathbf{H}^0$ can be TM- or TE-mode of this waveguide.

LECTURE 2

INTRODUCTION TO THE INTEGRAL EQUATION METHOD

The potential theory developed for the Laplace equation can be extended to the Helmholtz equation

$$\mathcal{L}(c)u := (\Delta + c)u = 0. \tag{180}$$

In order to construct fundamental solutions consider, in spherical coordinates, a solution $v_0 = v_0(r)$ depending only on r; the Laplace operator has the form

$$\Delta v_0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dv_0}{dr}\right) = \frac{1}{r}\frac{d^2(rv_0)}{dr^2}, \tag{181}$$

which yields an ordinary differential equation

$$\frac{d^2w}{dr^2} + cw = 0, \quad w = v_0 r. \tag{182}$$

Its linearly independent solutions are

$$\frac{e^{ikr}}{r}, \quad \frac{e^{-ikr}}{r} \quad (c = k^2 > 0), \tag{183}$$

$$\frac{e^{-\kappa r}}{r}, \quad \frac{e^{\kappa r}}{r} \quad (c = -\kappa^2 < 0). \tag{184}$$

The fundamental solution

$$\phi_0(r) = \frac{e^{-ikr}}{r} \tag{185}$$

corresponds to an outgoing spherical wave

$$u_0(r) = \frac{e^{i(\omega t - kr)}}{r} \tag{186}$$

propagating off a source placed in the origin $r = 0$ where $\phi_0(r)$ has a singularity $\sim \frac{1}{r}$.
Another solution

$$v_0(r) = \frac{e^{ikr}}{r} \tag{187}$$

corresponds to an incoming spherical wave

$$u_0(r) = \frac{e^{i(\omega t + kr)}}{r} \tag{188}$$

propagating from a source at infinity. This solution is ignored because it has no direct physical sense.

Using notation (180) we can write the second Green formula for the Helmholtz operator $\mathcal{L}$ and a domain T bounded by a piecewise smooth surface Σ

$$\int_T [u\mathcal{L}v - v\mathcal{L}u]d\tau = \int_\Sigma \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma. \tag{189}$$

Substituting instead of v a fundamental solution to the Helmholtz equation in the case of three dimensions and repeating literally the proof applied for obtaining an integral representation for a solution to the Poisson equation $\Delta u = -f$ (the third Green formula), we arrive at the integral representation of solution to the inhomogeneous Helmholtz equation $\mathcal{L}(k^2)u = -f$

$$
\begin{aligned}
u(\mathbf{r}) &= \frac{1}{4\pi} \int_\Sigma \left[\frac{e^{-ikR}}{R} \frac{\partial u}{\partial \nu} - u\frac{\partial}{\partial \nu}\left(\frac{e^{-ikR}}{R} \right) \right] d\sigma_{\mathbf{r}_0} + \frac{1}{4\pi} \int_T f(\mathbf{r}_0)\frac{e^{ikR}}{R} d\tau_{\mathbf{r}_0}, \\
R &= |\mathbf{r} - \mathbf{r}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}.
\end{aligned}
\tag{190}
$$

One can show that *the volume potentials*

$$v_1(\mathbf{r}) = \frac{1}{4\pi} \int_T f(\mathbf{r}_0)\frac{e^{-ikR}}{R} d\tau_{\mathbf{r}_0}, \quad v_2(\mathbf{r}) = \frac{1}{4\pi} \int_T f(\mathbf{r}_0)\frac{e^{ikR}}{R} d\tau_{\mathbf{r}_0} \tag{191}$$

satisfies the inhomogeneous Helmholtz equation $\mathcal{L}(k^2)u = -f$. However, both these functions decay at infinity. This fact dictates the necessity to introduce additional conditions specifying the behavior of solutions to the Helmholtz equation at infinity which would enable one to uniquely determine the solution.

Formulate the interior Dirichlet problem for the Helmholtz equation: find a function u continuous in $\bar{D} = D \cup \Gamma$ that satisfies the Helmholtz equation in a domain D bounded by the closed smooth contour Γ,

$$\mathcal{L}(k^2)u = \Delta u + k^2 u = 0 \quad \text{in} \quad D, \tag{192}$$

and the Dirichlet boundary condition

$$u|_\Gamma = -f, \tag{193}$$

where f is a given continuous function.

Formulate the interior Neumann problem: find a function u continuously differentiable in $\bar{D} = D \cup \Gamma$ that satisfies the Helmholtz equation (192) in domain D bounded by the closed smooth contour Γ and the Neumann boundary condition

$$\frac{\partial u}{\partial n}\bigg|_\Gamma = -g, \tag{194}$$

where $\frac{\partial}{\partial n}$ denotes the directional derivative in the direction of unit normal vector $\mathbf{n}$ to the boundary Γ directed into the exterior of Γ and g is a given continuous function.

Let us also formulate *Dirichlet and Neumann boundary eigenvalue problems for the Laplace equation*: find a nontrivial solution $u \in C(\bar{D})$ or, respectively, $u \in C^1(\bar{D})$ to the homogeneous Dirichlet or Neumann BVPs

$$-\Delta u = \lambda u \quad \text{in} \quad D, \quad u|_{\Gamma} = 0, \tag{195}$$

or

$$-\Delta u = \lambda u \quad \text{in} \quad D, \quad \left.\frac{\partial u}{\partial n}\right|_{\Gamma} = 0, \tag{196}$$

that correspond to certain (in general complex) values λ called *eigenvalues*.

It is known that eigenvalues of the Dirichlet and Neumann boundary eigenvalue problems for the Laplace equation in a domain D form the sets $\Lambda_{Dir,\,Neu} = \{\lambda_n^{D,N}\}_{n=1}^{\infty}$ of isolated real numbers $\lambda_n^{D,N}$ with the accumulation point at infinity; also, $0 \notin \Lambda_{Dir}$ and $0 \in \Lambda_{Neu}$. The complements $\rho_{Dir,\,Neu} = \mathrm{C} \setminus \Lambda_{Dir,\,Neu}$, where C denotes the complex λ-plane, are called resolvent (regular) sets of the Dirichlet or Neumann BVPs for the Laplace equation in D.

According to the definition, the (interior) Dirichlet or Neumann BVPs (192), (193) or (192), (194) for the Helmholtz equation in D have at most one solution if λ is not an eigenvalue; that is, if $\lambda \in \rho_{Dir}(D)$ or $\lambda \in \rho_{Neu}(D)$ is a regular value.

Theorem 26

Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The double layer potential

$$v(\mathbf{r}) = \int\limits_{\Gamma} \frac{\partial}{\partial n_{\mathbf{r_0}}} E(\mathbf{r} - \mathbf{r_0}) \varphi(\mathbf{r_0}) dl_{\mathbf{r_0}} \tag{197}$$

with a continuous density φ is a solution of the interior Dirichlet problem (192), (193) provided that $\lambda \in \rho_{Dir}(D)$ is a regular value and φ is a solution of the integral equation

$$\varphi(\mathbf{r}) - 2 \int\limits_{\Gamma} \frac{\partial E(\mathbf{r} - \mathbf{r_0})}{\partial n_{\mathbf{r_0}}} \varphi(\mathbf{r_0}) dl_{\mathbf{r_0}} = -2f(\mathbf{r}), \quad \mathbf{r} \in \Gamma. \tag{198}$$

Theorem 27

Let $D \in \mathrm{R}^2$ be a domain bounded by the closed smooth contour Γ. The single layer potential

$$u(\mathbf{r}) = \int_{\Gamma} E(\mathbf{r} - \mathbf{r}_0)\psi(\mathbf{r}_0)dl_{\mathbf{r}_0} \tag{199}$$

with a continuous density ψ is a solution of the interior Nuemann problem (192), (194) provided that $\lambda \in \rho_{Neu}(D)$ is a regular value and ψ is a solution of the integral equation

$$\psi(\mathbf{r}) + 2\int_{\Gamma} \frac{\partial E(\mathbf{r} - \mathbf{r}_0)}{\partial n_{\mathbf{r}}}\psi(\mathbf{r}_0)dl_{\mathbf{r}_0} = 2g(\mathbf{r}), \quad \mathbf{r} \in \Gamma. \tag{200}$$
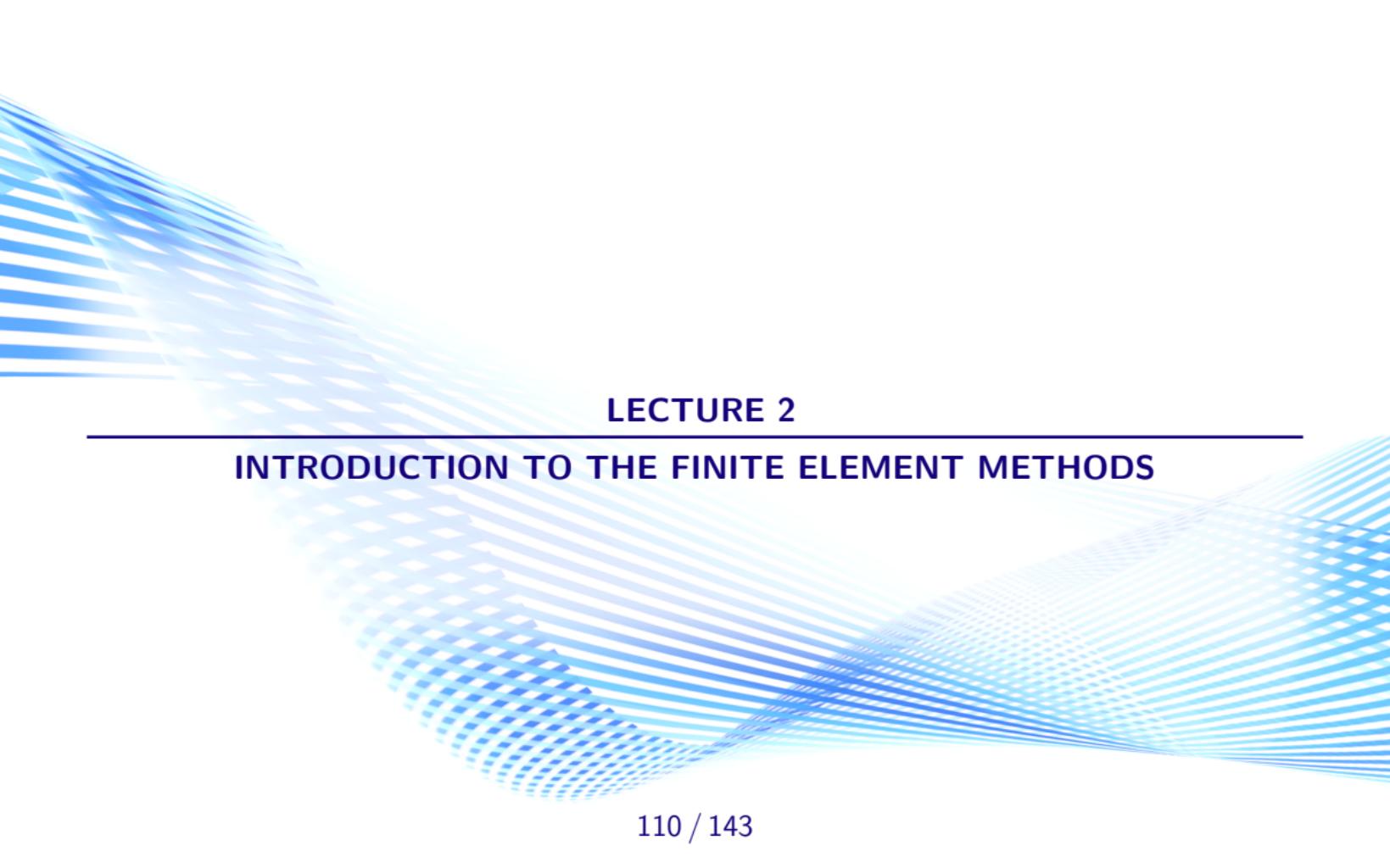
LECTURE 2

INTRODUCTION TO THE FINITE ELEMENT METHODS

Piecewise linear elements. An n-dimensional vector

$$\mathbf{a} = [a_1, a_2, \ldots, a_n]$$

is defined as an element of an n-dimensional space R^n and is an ordered set of n components $a_1, a_2, \ldots, a_n$. The n vectors

$$\mathbf{i_1} = [1, 0, 0, \ldots, 0], \quad \mathbf{i_2} = [0, 1, 0, \ldots, 0], \quad \ldots, \quad \mathbf{i_n} = [0, \ldots, 0, 1]. \tag{201}$$

form an (orthonormal) basis in R^n. Each vector $\mathsf{a} = \mathbf{a} = [a_1, a_2, \ldots, a_n] \in \mathsf{R}^n$ can be written as a linear combination of the basis vectors,

$$\mathbf{a} = a_1 \mathbf{i_1} + a_2 \mathbf{i_2} + \ldots, +a_n \mathbf{i_n}. \tag{202}$$

To introduce the piecewise linear finite elements, divide interval $[0, 1]$ in M (smaller) intervals $K_j = [x_{j-1}, x_j]$, $j = 1, 2, \ldots, M$ ($M \geq 2$), with the points

$$x_0 = 0 < x_1 < x_2 < \cdots < x_{M-1} < x_M = 1.$$

(in general, nonuniformly distributed with different distances between them $h_j = x_j - x_{j-1}$, $j = 1, 2, \ldots, M$). The corresponding $(M + 1)$-dimensional vector

$$\mathbf{X_M} = [x_0, x_1, x_2, \ldots, x_{M-1}, x_M] \tag{203}$$

is called *partition* of the base interval $[0, 1]$.

Note that for the points x_j uniformly distributed with the distance $h = \dfrac{1}{n}$,

$$
\begin{aligned}
x_j &= jh, \quad j = 0, 1 \dots, M, \\
x_0 &= 0 < x_1 = h < x_2 = 2h < \cdots < x_{M-1} = (M-1)h < x_M = Mh = 1.
\end{aligned}
\tag{204}
$$

The corresponding partition

$$
\mathbf{X_M} = [0, h, 2h, \dots, (M-1)h, 1] = h[0, 1, 2, \dots, M-1, M].
\tag{205}
$$

The piecewise linear elements are defined as

$$
\Phi_j(x) = \begin{cases}
0 & x_0 \leq x \leq x_{j-1}, \\
\frac{x - x_{j-1}}{h_j} & x_{j-1} \leq x \leq x_j, \\
\frac{x_{j+1} - x}{h_{j+1}} & x_j \leq x \leq x_{j+1}, \\
0 & x_{j+1} \leq x \leq x_M,
\end{cases}
\quad j = 1, 2 \dots M - 1,
$$

$$
h_j = x_j - x_{j-1}, \quad j = 1, 2 \dots, M.
\tag{206}
$$

Each $\Phi_j(x)$ is a piecewise linear 'rectangular' function such that

$$
\Phi_i(x_j) = \begin{cases}
1 & i = j, \\
0 & i \neq j;
\end{cases}
\quad , \quad i, j = 1, 2 \dots M - 1,
\tag{207}
$$

it does not equal 0 in each subinterval $[x_{j-1}, x_{j+1}] = K_j \cup K_{j+1}, \, j = 1, 2, \dots, M - 1$ (see Fig. .2).
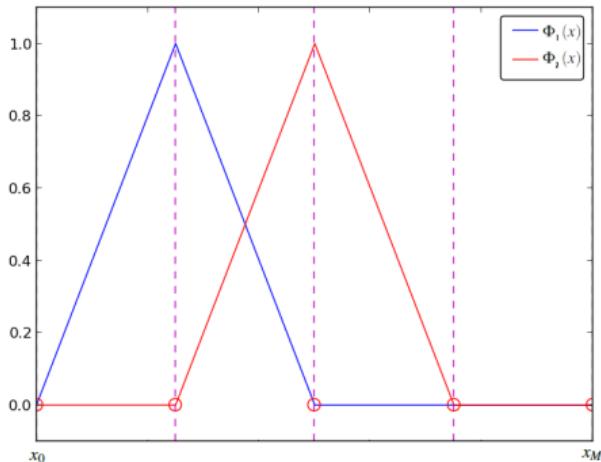
Figure .2: The piecewise linear elements.

Assume that $\mathbf{X_M} = [x_0, x_1, x_2, \ldots, x_{M-1}, x_M]$ is a given partition of $[0, 1]$ into M subinterval $K_j = [x_{j-1}, x_j]$, $j = 1, 2, \ldots, M$ $(M \geq 2)$. Define the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions

$$S_h = \{v \in S_h : \ v \text{ a linear in each subinterval } K_j, \ v(0) = v(1) = 0, \ h = \max h_j\}. \tag{208}$$

Theorem 28

The set $\{\Phi_j(x)\}$ of piecewise linear elements is a basis in space S_h; i.e., any piecewise linear function can be written as a linear combination of $\Phi_j(x)$.

Proof. A piecewise linear function $F = F(M; x)$ defined on the interval $[0, 1]$ is a linear function on each subinterval $K_j = [x_{j-1}, x_j]$, $j = 1, 2, \ldots, M$. This function vanishes on the endpoints of interval $[0, 1]$ so, that $F(M, 0) = F(M, 1) = 0$ and has $M - 1$ vertices and its derivative is undefined in these points.
Thus the function $F = F(M; x)$ is composed of M piecewise linear functions $F_j(x)$,

$$F(M; x) = \left\{ \begin{array}{ll} \ldots & \ldots, \\ F_j(x), & x \in K_j, \quad , \quad j = 1, 2 \ldots M, \\ \ldots & \ldots, \end{array} \right. \tag{209}$$

Function $F = F(M; x) \in S_h$ has values T_j in nodes x_j, $j = 1, 2, \ldots, M - 1$ (i.e., the function goes through the points (x_j, T_j), $j = 0, 1, 2, \ldots, M$) and for endpoints of the interval is defined as $F(M, 0) := T_0 = 0$, and $F(M, 1) := T_M = 0$, respectively. Thus any subfunction $F_j(x)$ goes through the points (x_{j-1}, T_{j-1}), (x_j, T_j), and uniquely determined on each subinterval $K_j = [x_{j-1}, x_j]$ (as a linear function) by

$$F_j(x_{j-1}) = T_{j-1}, \quad F_j(x_j) = T_j, \tag{210}$$

We obtain that any piecewise linear function $F = F(M; x) \in S_h$ which has values T_j in the nodes x_j is uniquely determined on the interval $[0, 1]$ under the conditions

$$F_j(x_j) = T_j, \quad j = 0, 1, 2, \ldots, M, \quad T_0 = T_M = 0. \tag{211}$$

Now let us show that any given piecewise linear function $F = F(M; x) \in S_h$ which has values T_j in the nodes x_j, $j = 0, 1, 2, \ldots, M$, with $T_0 = T_M = 0$ is a linear combination of piecewise linear base elements $\Phi_j(x)$. A linear combination of $\Phi_j(x)$ is

$$\tilde{F}(x) = \sum_{i=1}^{M-1} T_i \Phi_i(x). \tag{212}$$

$\tilde{F}(x)$ is a piecewise linear function (as a sum of piecewise linear functions) and

$$\tilde{F}(x_j) = \sum_{i=1}^{M-1} T_i \Phi_i(x_j) = T_j \tag{213}$$

$$\tilde{F}(x_0) = \sum_{i=1}^{M-1} T_i \Phi_i(x_0) = 0, \quad \tilde{F}(x_M) = \sum_{i=1}^{M-1} T_i \Phi_i(x_0) = 0, \tag{214}$$

according to (207), so

$$\tilde{F}(x) = \sum_{i=1}^{M-1} T_i \Phi_i(x) = F(M; x) \in S_h. \tag{215}$$

Consider a $(M-1)$-dimensional space $S_h = (\mathbf{X_m})$ of piecewise linear functions. The minimal value of parameter $M = 2$ gives us two subintervals $K_1 = [x_0, x_1]$ and $K_2 = [x_1, x_2]$; Then the corresponding partition

$$\mathbf{X_2} = [x_0, x_1, x_2] = [0, x_1, 1] \tag{216}$$

is a 3-dimensional vector. For this partition, we can define only piecewise linear 'triangular' elements $\Phi_1(x)$ by formula (206) for $j = 1$

$$\Phi_1(x) = \begin{cases} \frac{x-x_0}{h_1} = \frac{x}{h_1} = \frac{x}{x_1} & 0 = x_0 \le x \le x_1, \\ \frac{x_2-x}{h_2} = \frac{1-x}{h_2} = \frac{1-x}{1-x_1} & x_1 \le x \le x_2 = 1, \\ h_1 = x_1 - x_0 = x_1, \quad h_2 = x_2 - x_1 = 1 - x_1, \end{cases} \tag{217}$$

which satisfies (according to (207))

$$\Phi_1(x_1) = 1, \quad \Phi_1(x_0) = \Phi_1(0) = 0, \quad \Phi_1(x_2) = \Phi_1(1) = 0, \tag{218}$$

and not equal to 0 on the whole interval $[x_0; x_2] = K_1 \cup K_2 = [0, 1]$. In this case $M = 2$ and the basic element $\Phi_1(x)$ (217) is an element of one-dimensional space $S_h = (\mathbf{x_2})$ which consists of one piecewise linear 'triangular' functions $v(x) := C\Phi_1(x)$ with an arbitrary C:

$$S_h = S_h(\mathbf{X_2}) = \{C\Phi_1(x) \quad \forall C \in \mathsf{R}\},$$

$$v(x) \in S_h(\mathbf{X_2}): \ v(x_1) = C, \quad v(x_0) = v(0) = 0, \quad v(x_2) = v(1) = 0. \tag{219}$$

LECTURE 2: INTRODUCTION TO THE FINITE ELEMENT METHODS.

In the same way one can show that in the case $M > 2$, the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ consisting of piecewise linear functions which take values T_j in nodes x_j, $j = 1, 2, \ldots, M-1$ and vanishes in nodes $x_0 = 0$ and $x_M = 1$ (and can be written as (212)),

$$S_h = S_h(\mathbf{X_M}) = \left\{ \sum_{i=1}^{M-1} T_i \Phi_1(x), \quad \forall \mathbf{T_M} = [T_1, T_2, \ldots, T_{M-1}] \right\}. \tag{220}$$

We can determine piecewise linear base elements $\Phi_j(x) \in S_h(\mathbf{X_M})$ with the base vectors (201) and a piecewise linear function $F = F(M; x) \in S_h(\mathbf{X_M})$ which takes values T_j, $T_0 = T_M = 0$, in nodes x_j, $j = 0, 1, 2, \ldots, M-1, M$. The $(M-1)$-dimensional vector of the values is

$$\mathbf{T_M} = [T_1, T_2, \ldots, T_{M-1}] \tag{221}$$

The set $C_0^1(\bar{I}_0)$ denotes a set of continuously differentiable in the closed interval $\bar{I}_0 = [0, 1]$ functions $f(x)$ which satisfy the following boundary conditions

$$f(x) \in C_0^1(\bar{I}_0): \quad f(0) = 0, \ f(1) = 0. \tag{222}$$

A projection $P_M(f)$ of a function $f(x) \in C_0^1(\bar{I}_0)$ in the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions with respect to a given partition (203) $\mathbf{X_M} = [x_0, x_1, \ldots, x_M]$ $(M \geq 2)$ is defined as (212)

$$P_M(f) = \sum_{i=1}^{M-1} f(x_i) \Phi_i(x). \tag{223}$$

We can determine the projection $P_M(f)$ as a $(M-1)$-dimensional vector

$$\mathbf{P_M} = [f_1, f_2, \ldots, f_{M-1}], \quad f_i = f(x_i). \tag{224}$$

Consider a **BVP for a linear differential equation of the second order**

$$
\begin{cases}
Ay = -(ay')' + q(x)y = f(x), & x \in I_0 = (0,1), \\
y(0) = 0, \quad y(1) = 0,
\end{cases}
\tag{225}
$$

where $a(x)$, $q(x)$ and $f(x)$ are smooth functions satisfying the following conditions

$$
a(x) \geq a_0 > 0, \quad q(x) \geq 0.
$$

The Variation formulation of BVP (225), or weak formulation, is given by

$$a(y, \phi) = (f, \phi) \quad \forall \phi \in C_0^1(\bar{I}_0), \tag{226}$$

where

$$a(y, \phi) = \int_0^1 [a(x)y'\phi' + q(x)y(x)\phi(x)]dx, \tag{227}$$

$$(f, \phi) = \int_0^1 f(x)\phi(x)dx. \tag{228}$$

Divide an interval $[0, 1]$ into M subintervals $K_j = [x_{j-1}, x_j]$ $j = 1, 2, \ldots, M$. The corresponding partition $\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M]$ ($M \geq 2$). To implement the numerical method for solving BVP (225), written in the weak form as integral equation (226), we replace functions $a(x)$, $y(x)$, $q(x)$ and $f(x)$, for $x \in \bar{I}_0 = [0, 1]$, with their projections in the $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ by piecewise linear functions with respect to a given partition $\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M]$ ($M \geq 2$) and (226) with a finite-dimensional approximation based on piecewise linear finite element (206).

Finite-dimensional approximation. Formulate a finite-dimensional problem which approximates BVP (225) or (226): find $u_h \in S_h(\mathbf{x_m})$ such that

$$a(u_h, \phi_h) = (f, \phi_h) \quad \forall \phi \in S_h(\mathbf{X_M}). \tag{229}$$

Here u_h is given by

$$u_h = \sum_{j=1}^{M-1} U_j \Phi_j(x); \tag{230}$$

it can be considered as the projection (223)

$$P_M(u) = \sum_{i=1}^{M-1} u(x_i) \Phi_i(x) \tag{231}$$

of the unknown solution $u(x) \in C_0^1(\bar{I}_0)$ of BVP (225) in $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions with respect to partition (203)

$$\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M] \quad (M \geq 2).$$

Insert (230) into (229) to find that (229) is equivalent to

$$\sum_{j=1}^{M-1} U_i a(\Phi_j(x), \Phi_i(x)) = (f, \Phi_i), \quad i = 1, 2, \ldots, M-1. \tag{232}$$

or in the matrix form

$$A\mathbf{U_M} = \mathbf{f}, \tag{233}$$

where vector $\mathbf{f} = [f_1, f_2, \ldots, f_{M-1}]$ the load vector,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,M-1} \\ a_{21} & a_{22} & \ldots & a_{2,M-1} \\ . & . & \ldots & . \\ a_{M-1,1} & a_{M-1,2} & \ldots & a_{M-1,M-1} \end{bmatrix},$$

or

$$A = \begin{bmatrix} a(\Phi_1, \Phi_1) & a(\Phi_1, \Phi_2) & a(\Phi_1, \Phi_3) & \ldots & a(\Phi_1, \Phi_{M-1}) \\ a(\Phi_2, \Phi_1) & a(\Phi_2, \Phi_2) & a(\Phi_2, \Phi_3) & \ldots & a(\Phi_2, \Phi_{M-1}) \\ a(\Phi_3, \Phi_1) & a(\Phi_3, \Phi_2) & a(\Phi_3, \Phi_3) & \ldots & a(\Phi_3, \Phi_{M-1}) \\ . & . & . & \ldots & . \\ a(\Phi_{M-1}, \Phi_1) & a(\Phi_{M-1}, \Phi_2) & a(\Phi_{M-1}, \Phi_3) & \ldots & a(\Phi_{M-1}, \Phi_{M-1}) \end{bmatrix} \tag{234}$$

is a *stiffness matrix*. The size (dimension) of matrix A is equal to $(M-1) \times (M-1)$ and it is a symmetric matrix: $a_{ij} = a_{ji}$.

Note that function $\Phi_j(x)$ vanishes at the endpoints of the interval. The elements of the stiffness matrix A of the BVP is determined by

$$a(\Phi_j(x), \Phi_i(x))) = a(\Phi_i(x), \Phi_j(x))) = \int_0^1 [\Phi_i'\Phi_j' + q(x)\Phi_i(x)\Phi_j(x)\phi(x)]dx \tag{235}$$

Some expressions $\Phi_i\Phi_j$ and $\Phi_i'\Phi_j'$ vanish, e.g.

$$(\Phi_j(x), \Phi_i(x)) = (\Phi_i(x), \Phi_j(x)) = \int_0^1 \Phi_j(x)\Phi_i(x)dx = 0, \quad |i-j| \geq 2 \ (i,j = 1, 2, \ldots, M-1). \tag{236}$$

Stiffness matrix A (234) of BVP (225) is a symmetric diagonal matrix of size $(M-1) \times (M-1)$

$$\mathcal{A} = \begin{bmatrix} c_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a & -b & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & -b & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & -b & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & -b & a \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & c_N \end{bmatrix} \tag{237}$$

with elements

$$a_{ij} = 0, \quad |i-j| \geq 2 \quad (i,j = 1,2,\ldots,M-1); \tag{238}$$

namely

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{M-2,M-3} & a_{M-2,M-2} & a_{M-2,M-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{M-1,M-2} & a_{M-1,M-1} \end{bmatrix} \tag{239}$$

If $q = const$, we get

$$
\begin{aligned}
a(\Phi_j(x), \Phi_i(x)) &= a(\Phi_i(x), \Phi_j(x)) = \int_0^1 [\Phi_i' \Phi_j' + q\Phi_i(x)\Phi_j(x)\phi(x)]dx = \\
&= (\Phi_i', \Phi_j') + q(\Phi_i, \Phi_j)
\end{aligned}
\tag{240}
$$

and we can rewrite stiffness matrix A (234) as a matrix sum

$$
A = Q_1 + Q_0, \quad Q_1 = \left[(\Phi_i', \Phi_j') \right], \quad Q_0 = q\left[(\Phi_i, \Phi_j) \right].
\tag{241}
$$

Consider an important case of the uniform partition (204) when points $x_j = jh$, $j = 0, 1 \ldots, M$, are distributed uniformly with the step $h = \dfrac{1}{M}$ and piecewise linear base elements are determined as

$$\Phi_j(x) = \begin{cases} 0 & x_0 \leq x \leq x_{j-1}, \\ \frac{x - x_{j-1}}{h} & x_{j-1} \leq x \leq x_j, \\ \frac{x_{j+1} - x}{h} & x_j \leq x \leq x_{j+1}, \\ 0 & x_{j+1} \leq x \leq x_M, \end{cases} \qquad j = 1, 2 \ldots M - 1. \tag{242}$$

The expressions $(\Phi_j(x), \Phi_i(x)) = 0$, $|i - j| \geq 2$ vanish according to (236). Nonzero elements are

$$(\Phi'_j, \Phi'_j) = \int_{x_{j-1}}^{x_j} \frac{1}{h^2} dx + \int_{x_j}^{x_{j+1}} \frac{1}{h^2} dx = \frac{2}{h}, \tag{243}$$

$$(\Phi'_{j-1}, \Phi'_j) = \int_{x_{j-1}}^{x_j} \frac{1}{h} \left(-\frac{1}{h} \right) dx = -\frac{1}{h}, \tag{244}$$

$$(\Phi_j, \Phi_j) = \int_{x_{j-1}}^{x_j} \frac{(x - x_{j-1})^2}{h^2} dx + \int_{x_j}^{x_{j+1}} \frac{(x_{j+1} - x)^2}{h^2} dx = \frac{2h}{3}, \tag{245}$$

$$(\Phi_{j-1}, \Phi_j) = \int_{x_{j-1}}^{x_j} \frac{(x - x_{j-1})}{h} \frac{(x_j - x)}{h} dx = \frac{h}{6}, \tag{246}$$

$(i, j = 1, 2, \ldots, M - 1)$.

Constant coefficients. If $q = const$, then the stiffness matrix A is rewritten as a sum of tridiagonal matrices

$$Q_1 = [(\Phi_i', \Phi_j')] = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix} \tag{247}$$

and

$$Q_0 = [(\Phi_i, \Phi_j)] = q\frac{h}{6} \begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 \end{bmatrix} \tag{248}$$

If $q = 0$, then the stiffness matrix A coincides with matrix Q_1. The finite-dimensional problem (225) approximates the following BVP

$$\begin{cases} -y'' = f(x), & x \in I_0 = (0,1), \\ y(0) = 0, \quad y(1) = 0, \end{cases} \tag{249}$$

or

$$A\mathbf{U_M} = \mathbf{f},$$

where

$$A = Q_1 = \left[(\Phi_i', \Phi_j')\right] = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \tag{250}$$

which coincides with the system obtained for BVP (249)

$$y'' - q(x)y = f(x), \ d_1 < x < d_2, \\ y(d_1) = f_0, \ y(d_2) = f_N. \tag{251}$$

The forward and backward differences are determined as

$$\Delta y_i = y_{x,i} = \frac{y_{i+1} - y_i}{h}, \ \nabla y_i = y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h} \tag{252}$$

and

$$y'' \approx y_{\bar{x}x} = y_{\bar{x}x,i} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \tag{253}$$

$$-y'' \approx -y_{\bar{x}x} = -y_{\bar{x}x,i} = \frac{-y_{i+1} + 2y_i - y_{i-1}}{h^2},$$

and

$$-y_{\bar{x}x,i} = f_i, \ i = 1, 2, \ldots, N-1, \qquad y_0 = 0, \ y_N = 0,$$

or

$$\begin{cases} y_0 &= 0 \\ \frac{1}{h}(-y_{i-1} + 2y_i - y_{i+1}) &= hf_i, \quad 1 \le i \le N-1, \\ y_N &= 0. \end{cases} \tag{254}$$

(254) is a system of linear equations.

Approximate solution. Let $\mathbf{U} = [U_1, \ldots, U_{M-1}]$ denote the solution of the linear system (233). The approximate solution of BVP (225) is defined as a solution of corresponding finite-dimensional problem (225)

$$u_h(x) = \sum_{j=1}^{M-1} U_j \Phi_j(x) \tag{255}$$

The approximate solution u_h is an element of $(M-1)$-dimensional space $S_h = S_h(\mathbf{X_M})$ of piecewise linear functions with respect to partition (203) $\mathbf{X_M} = [x_0, x_1, \ldots, x_{M-1}, x_M]$ ($M \geq 2$).

Error estimate The error $r = r(h)$ of the approximate solution of BVP (225) can be defined as

$$r(h) = ||u_h - y||_2 = \sqrt{\int_0^1 [u_h(x) - y(x)]^2}, \tag{256}$$

where $y(x)$ represents the exact solution of BVP (225) and h is the maximum length between adjacent nodes. The error can be calculated approximately as the Euclidean norm

$$r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_2 = \sqrt{\sum_{j=1}^{M-1} (U_j - y_j)^2}, \tag{257}$$

i.e., length of the discrepancy vector $\mathbf{U_M} - \mathbf{Y_M}$, where $\mathbf{Y_M} = [y_1, \ldots, y_{M-1}]$ with $y_j = y(x_j)$, $j = 1, 2, \ldots M - 1$, is the projection (231) $P_M(y) = \sum_{i=1}^{M-1} y(x_i)\Phi_i(x)$ of the sought for solution $y(x) \in C_0^1(\bar{I}_0)$.

One can also determine the error approximately with the help of the maximum norm

$$r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_c = \max_{1 \le j \le M-1} |U_j - y_j|. \tag{258}$$

One can show that the following estimates hold to the relative error

$$\frac{||u_h - y||_2}{||y||_2} \le Ch^2 \tag{259}$$

with some constant C. This means that one can solve approximately the BVP using the finite element method with sufficiently small step h.

Example 12

Solve the BVP

$$\begin{cases} -y'' + 4y = 2, & 0 < x < 1, \\ y(0) = 0, \quad y(1) = 0 \end{cases} \tag{260}$$

with the help of the finite element method (uniform partition) by reducing it to a system of linear equations with three unknowns. Calculate the approximate solution u_h and determine the (approximate) error $||u_h - y||$ where $y(x)$ is an exact solution of (260).

Solution. The weak formulation of BVP(260) is

$$a(y, \phi) = (f, \phi) \quad \forall \phi \in C_0^1(\bar{I}_0), \tag{261}$$

where

$$a(y, \phi) = \int_0^1 [y'\phi' + 4y(x)\phi(x)]dx = \int_0^1 y'\phi' dx + 4\int_0^1 y(x)\phi(x)dx = (y', \phi') + 4(y, \phi), \tag{262}$$

$$(f, \phi) = 2\int_0^1 \phi(x)dx. \tag{263}$$

The finite-dimensional problem, which approximates BVP (260) or equal weak problem (261), is reduced to a system of linear equations with $M - 1 = 3$ unknowns U_1, U_2, U_3 and three equations. In the case $M = 4$ we obtain four subintervals

$$
\begin{array}{rcl}
K_1 & = & [x_0, x_1] = [0, h], \quad K_2 = [x_1, x_2] = [h, 2h], \\
K_3 & = & [x_2, x_3] = [2h, 3h], \quad K_4 = [x_3, x_4] = [3h, 4h] = [3h, 1]
\end{array} \tag{264}
$$

with uniform partition

$$
\mathbf{X_4} = [x_0, x_1, x_2, x_3, x_4] = [0, x_1, x_2, x_3, 1] = [0, h, 2h, 3h, 4h] = h[0, 1, 2, 3, 4], \quad h = 0.25. \tag{265}
$$

For this partition, we can define a piecewise linear base element $\Phi_j(x)$ according to (206) with $j = 1, 2, 3$. The linear system of equation $A\mathbf{U} = \mathbf{f}$ with three unknowns approximates BVP (260).

The tridiagonal stiffness matrix A has a size 3×3. We have $q = const = 4$; the stiffness matrix A is a sum of symmetric tridiagonal matrices

$$Q_1 = \left[(\Phi_i', \Phi_j') \right]_{i,j=1}^3 = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}, \tag{266}$$

$$Q_0 = [(\Phi_i, \Phi_j)] = 4\frac{h}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix}, \tag{267}$$

and

$$\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} = \left[(\Phi_i', \Phi_j') + 4(\Phi_i, \Phi_j) \right] = Q_1 + Q_0 = \\
&= \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix} + \begin{bmatrix} 2/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 2/3 \end{bmatrix} = \begin{bmatrix} 26/3 & -23/6 & 0 \\ -23/6 & 26/3 & -23/6 \\ 0 & -23/6 & 26/3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 52 & -23 & 0 \\ -23 & 52 & -23 \\ 0 & -23 & 52 \end{bmatrix}
\end{aligned}$$

The right side of system is determined as (263)

$$f_i = (f, \Phi_i) = 2\int_0^1 \Phi_i(x)dx = 2\int_0^1 \Phi_i(x)dx = 2\int_{x_{i-1}}^{x_{i+1}} \Phi_i(x)dx = 2h = 0.5, \quad i = 1, 2, 3.$$

Now we can write the linear system (**??**) $A\mathbf{U} = \mathbf{f}$ with three unknowns which approximates (260)

$$
\begin{array}{rcl}
52U_1 - 23U_2 & = & 3, \\
-23U_1 + 52U_2 - 23U_3 & = & 3 \\
-23U_2 + 52U_3 & = & 3
\end{array}
\tag{268}
$$

(we multiply both sides by 6).

Solve this system using the Gaussian elimination:

$$23U_1 - (23^2/52)U_2 = 3(23/52),$$
$$-23U_1 + 52U_2 - 23U_3 = 3$$
$$-23U_2 + 52U_3 = 3$$

$$23U_1 - (23^2/52)U_2 = 3(23/52),$$
$$(52 - (23^2/52))U_2 - 23U_3 = 3(1 + (23/52))$$
$$-23U_2 + 52U_3 = 3$$

$$23U_1 - (23^2/52)U_2 = 3(23/52),$$
$$23U_2 - 23^2/(52 - (23^2/52))U_3 = 3 \cdot 23(1 + (23/52))/(52 - (23^2/52))$$
$$-23U_2 + 52U_3 = 3$$

$$23U_1 - (23^2/52)U_2 = 3(23/52),$$
$$23U_2 - 23^2/(52 - (23^2/52))U_3 = 3 \cdot 23(1 + (23/52))/(52 - (23^2/52))$$
$$(52 - 23^2/(52 - (23^2/52)))U_3 = 3 + 3 \cdot 23(1 + (23/52))/(52 - (23^2/52))$$

The solution of system (268) is

$$U_1 = \frac{225}{1646} = 0.137$$
$$U_2 = \frac{294}{1646} = 0.179$$
$$U_3 = \frac{225}{1646} = 0.137$$

The error is approximately calculated using the Euclidean norm (257)

$$r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_2 = \sqrt{\sum_{j=1}^{3}(U_j - y_j)^2}. \tag{269}$$

The exact solution of (260) is

$$
\begin{aligned}
y(x) &= Ae^{2x} + Be^{-2x} + \frac{1}{2}, \tag{270} \\
A &= \frac{1}{2}\frac{1 - e^{-2}}{e^{-2} - e^2} = -0.060, \\
B &= \frac{1}{2}\frac{e^2 - 1}{e^{-2} - e^2} = -0.440. \tag{271}
\end{aligned}
$$

Projection (223) $P_M(f) = \sum_{i=1}^{3} y(x_i)\Phi_i(x)$ can be identified with 3-dimensional vector (224)

$$
\begin{aligned}
\mathbf{Y_M} &= [y_1, y_2, y_3], \\
y_i &= y(x_i) = y(ih) = y\left(\frac{i}{4}\right) = Ae^{i/2} + Be^{-i/2} + \frac{1}{2}, \ i = 1, 2, 3.
\end{aligned}
$$

We have

$$
\begin{array}{rcl}
y_1 &=& 0.5 - 0.06e^{1/2} - 0.44e^{-1/2} = 0.133, \\
y_2 &=& 0.5 - 0.06e - 0.44e^{-1} = 0.175, \\
y_1 &=& 0.5 - 0.06e^{3/2} - 0.44e^{-3/2} = 0.133.
\end{array}
$$

The target error

$$
\begin{array}{rcl}
r(h) &\approx& ||\mathbf{U_M} - \mathbf{Y_M}||_2 = \sqrt{\sum_{j=1}^{3}(U_j - y_j)^2} = \\
&=& \sqrt{(0.137 - 0.133)^2 + (0.179 - 0.175)^2 + (0.137 - 0.133)^2} = \\
&=& \sqrt{3 \cdot 0.004^2} = \sqrt{12 \cdot 10^{-6}} = 3.464 \cdot 10^{-3} \approx 0.003.
\end{array}
\tag{272}
$$

The error can also be determined approximately with the help of the maximum norm (258)

$$
r(h) \approx ||\mathbf{U_M} - \mathbf{Y_M}||_c = \max_{1 \leq j \leq 3} |U_j - y_j| = 0.004.
\tag{273}
$$
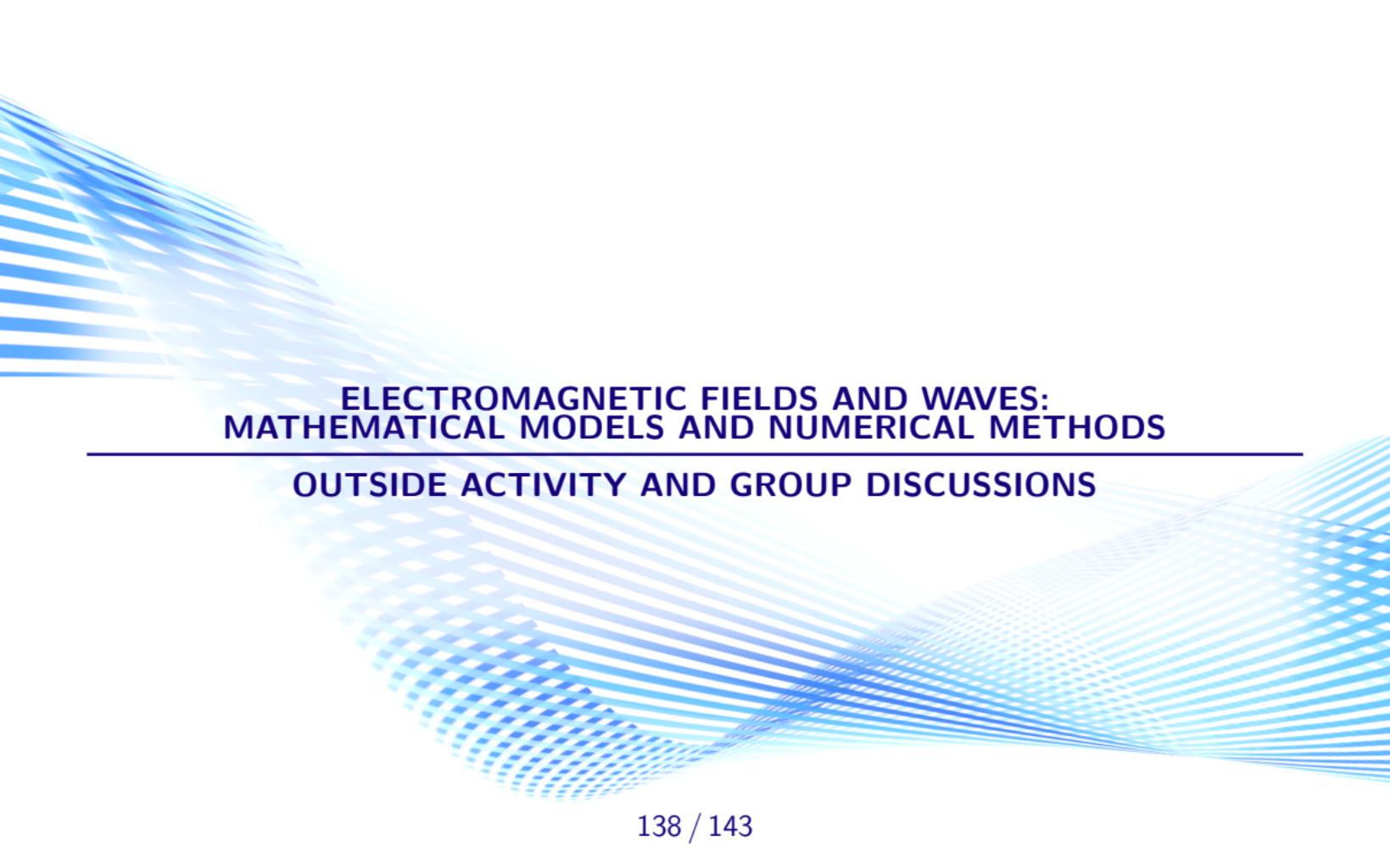
ELECTROMAGNETIC FIELDS AND WAVES: MATHEMATICAL MODELS AND NUMERICAL METHODS

OUTSIDE ACTIVITY AND GROUP DISCUSSIONS

Problem 1 Find derivative $\dfrac{dw}{dt}$ of the function $w = \sqrt{x^2 + y^2}$ where $x = e^{4t}$ and $y = e^{-4t}$.

Problem 2 Find grad f of the function $f(x,y) = x^2 - y^2$ and its value and length at the point $P : (-1,3)$.

Problem 3 Find the gradient $-\text{grad} f$ for $f(x,y,z) = z/(x^2 + y^2)$ and its value at the point $P : (0,1,2)$.

Problem 4 Determine the divergence of

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

Problem 5 Find curl of the vector field

$$\mathbf{v} = \frac{1}{2}(x^2 + y^2 + z^2)(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Problem 6 Determine a normal vector and unit normal vector to the xy-plane

$$\mathbf{r}(u,v) = [u,v] = u\mathbf{i} + v\mathbf{j}$$

and parametric form of curves $u = const$ and $v = const$.

Problem 7 Prove that the function (8)

$$\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{y}|}$$

(the fundamental solution of the Laplace equation) is harmonic with respect to the coordinates of $\mathbf{x}$ for a fixed $\mathbf{y} \in \mathrm{R}^2$, $\mathbf{y} \neq \mathbf{x}$ and with respect to $\mathbf{y}$ for a fixed $\mathbf{x} \in \mathrm{R}^2$, $\mathbf{x} \neq \mathbf{y}$.

Problem 8 Reduce to a boundary integral equation the BVP in a rectangle
$\Pi_{ab} = \{(x, y) : 0 < x < a, 0 < y < b\}$

$$\begin{cases} \Delta u = 0, \quad u = u(x, y), \quad 0 < x < a, \ 0 < y < b, \quad u \in C^2(\Pi_{ab}) \cap C(\bar{\Pi}_{ab}) \\ u(0, y) = 0, \quad u(a, y) = 0, \quad 0 \leq y \leq b, \\ u(x, 0) = 0, \quad u(x, b) = H(x), \quad 0 \leq x \leq a, \end{cases}$$

$$H(x) = \begin{cases} Q[p^2 - (x - x_S)^2]^2 e^{-r(x - x_S)^2}, \ |x - x_S| \leq p, \\ 0, \quad |x - x_S| \geq p, \end{cases} \tag{274}$$

with $supp\, H(x) = L = (x_S - p, x_S + p) \subset (0, a)$.

Problem 9 Let

$$\mathbf{E} = \operatorname{grad} \operatorname{div} \mathbf{P} + k^2 \mathbf{P}, \quad \mathbf{H} = -ik \operatorname{rot} \mathbf{P}, \quad \mathbf{P} = [0, 0, \Pi].$$

Problem 10 Apply separation of variables and find eigenvalues λ_n^D and eigenfunctions of the Dirichlet boundary eigenvalue problem (195) for the Laplace equation in a rectangle Π_{ab} (see problem 8). Determine normalized eigenfunctions with respect to the norm generated by the inner product $(f, g) = \displaystyle\int \int_{\Pi_{ab}} f g \, dx dy$ in the space $L_2(\Pi_{ab})$ of square-integrable functions.

Miniproject: example of inverse problem Prove that in the BVP in a rectangle
$\Pi_{ab} = \{(x,y): 0 < x < a, 0 < y < b\}$

$$
\begin{cases}
-\Delta u = F(x,y), & u = u(x,y), \quad 0 < x < a, \ 0 < y < b, \qquad u \in C^2(\Pi_{ab}) \cap C(\bar{\Pi}_{ab}) \\
u(0,y) = 0, & u(a,y) = 0, \quad 0 \le y \le b, \\
u(x,0) = 0, & u(x,b) = 0, \quad 0 \le x \le a,
\end{cases}
$$

$$
F(x,y) = \begin{cases}
A \sin \frac{\pi}{h_1} \left[x - \left(x_0 - \frac{h_1}{2} \right) \right] \sin \frac{\pi}{h_2} \left[y - \left(y_0 - \frac{h_2}{2} \right) \right], & (x,y) \in \Pi_{h_1 h_2}(x_0, y_0), \\
0, & (x,y) \notin \Pi_{h_1 h_2}(x_0, y_0),
\end{cases}
\tag{275}
$$

with $\quad supp\, F(x,y) = \Pi_{h_1 h_2}(x_0, y_0) \subset \Pi_{ab},$

$$
\Pi_{h_1 h_2}(x_0, y_0) = \left\{ (x,y): x_0 - \frac{h_1}{2} < x < x_0 + \frac{h_1}{2}, \ y_0 - \frac{h_2}{2} < y < y_0 + \frac{h_2}{2} \right\}.
$$

it is possible, under certain conditions, to uniquely determine any of the five parameters A, x_0, y_0, h_1, h_2 provided that the remaining four are given from the knowledge of one Fourier coefficient $u_1 = u_1(A, x_0, y_0, h_1, h_2)$ of $u(x,y)$.

Problem 11 Determine explicit expressions for TM-waves in a waveguide of rectangular cross section
$\Pi_{ab} = \{(x_1, x_2) : 0 < x_1 < a, \, 0 < x_2 < b\}$.

Problem 12 The normal wave propagating along x_3-axis in a waveguide with cross section Ω that corresponds to
the first (minimal) eigenvalue λ_1 of the Dirichlet boundary eigenvalue problem for the Laplace equation in Ω is
often called *the fundamental TM mode* of the waveguide. Determine an explicit expression for the fundamental
TM mode in a waveguide of rectangular cross section $\Omega = \Pi_{ab} = \{(x_1, x_2) : 0 < x_1 < a, \, 0 < x_2 < b\}$.